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Rotating membranes on G_2 manifolds, logarithmic anomalous dimensions and $N = 1$ duality

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Abstract: We show that the $E - S \sim \log S$ behaviour found for long strings rotating on $AdS_5 \times S^5$ may be reproduced by membranes rotating on $AdS_4 \times S^7$ and on a warped AdS_5 M-theory solution. We go on to obtain rotating membrane configurations with the same $E - K \sim \log K$ relation on G_2 holonomy backgrounds that are dual to $\mathcal{N} = 1$ gauge theories in four dimensions. We study membrane configurations on G_2 holonomy backgrounds systematically, finding various other Energy-Charge relations. We end with some comments about strings rotating on warped backgrounds.

Keywords: M-Theory, AdS-CFT and dS-CFT Correspondence.

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1. Background and motivation

1.1 A brief history of rotating strings and membranes

A recent advance in our understanding of the AdS/CFT duality was the proposal [1] that gauge theory operators with large spin were dual to semiclassical rotating strings in the AdS background. This original work was inspired by comments [2] concerning 'long' gauge theory operators with high bare dimension and by the success in matching the anomalous dimensions of large R-charge operators with the spectrum of string theory on the Penrose limit of $AdS_5 \times S^5$ [3] . String configurations naturally have energies in the $1/\alpha' \sim \lambda^{1/2}$ scale, where λ is the 't Hooft coupling, and are therefore dual to operators with large dimensions. Rotating strings [1] were shown to reproduce the known results for large R-charge operators [3], as well as giving results for a new class of 'long' twist two operators.

The principle factor that made the identification of these new operators possible, was the fact that a rotating string configuration in AdS space was obtained [1] that had $E-S \sim f(\lambda)\ln S$, where E and S are the energy and spin of the configuration. These must then be dual to gauge theory operators with an anomalous dimension that depends logarithmically on the spin. Such operators were known from the operator approach to Deep Inelastic Scattering (D.I.S.) in QCD, where they appear in the OPE of electromagnetic currents. The twist two operators typically have the form

$$\mathcal{O}_S(x) = \Phi(x)\nabla_{\mu_1}\cdots\nabla_{\mu_S}\Phi(0)\,. \tag{1.1}$$

Where $\Phi(x)$ is a field in the theory such as a field strength or quarks. The anomalous dimension of these twist two operators is responsible for violations of Bjorken scaling in D.I.S. at finite coupling [4]. It is perhaps surprising that the logarithmic dependence of anomalous dimension on spin survives from the perturbative to the strong coupling regime in the 't Hooft coupling, and that no corrections of $\ln^k S$, or other corrections, appear. Some corrections were shown to vanish in an important one loop string calculation [5]. These results [5] further clarified the connection with the large R-charge operators.

The work described so far [1] has subsequently been developed on both the gauge theory and string theory side in several works [5]–[16]. Let us summarise briefly these works.

In [5, 6], solutions interpolating smoothly between the various configurations considered in [1] were found and a natural proposal for some of the dual operators was given. In [11], non-stationary, pulsating, configurations were considered and, using a WKB approximation, the corrected energies were found in the limit of large quantum numbers. These configurations were associated with generalisations of operators with impurities studied in [3]. In [9], orbifolded geometries were considered using a collective coordinates approach, and some connections between gauge theories and integrable systems were pointed out. The paper [7] considers strings orbiting around AdS_5 black holes. Their proposal is to understand the finite temperature dual system as a glueball melting into the gluon plasma, due to a transfer of angular momentum from the 'planetoid' solution [17], to the black hole.

In [8], an interesting extension was proposed. Firstly, they analyse the behaviour expected when one considers string configurations in 'confining geometries', considering only the general features of this type of geometry. They propose that the functional form has to change, when varying the size of the string soliton, from Regge-like ($E \sim S^{1/2}$) to D.I.S-like ($E - S \sim \ln S$). A key observation of theirs is that in this case, unlike the case of $AdS_5 \times S^5$, the Regge-like behaviour will not be simply an effect of the finite volume in which the gauge theory dual is defined. Further, they study string solitons in Witten's model for QCD [18] and find, for a model constructed with near extremal Dp-branes, a very curious relation of the form $E - S \sim 1 - S^{(p-5)/(9-p)}$. One result of our work will be to exhibit a Regge to logarithmic transition without any finite size effects.

In [15], the study of strings in Witten's model was extended by considering pulsating, non-stationary, configurations similar to those of [11]. Further, [15] study pulsating membrane configurations in AdS_7. In [13], rotating membranes were studied in AdS_7 spaces, but no logarithmic behaviour for the anomalous dimensions was found. Instead, power-like behaviours were displayed. In a very nice paper dealing with higher spin gauge theories [16], the authors also discuss membrane configurations and found similar results. Another result of the present paper is to obtain a logarithmic anomalous dimension for membranes in AdS_4 (or AdS_7).

More recently, the results of [1] were reproduced using Wilson loops with a cusp anomaly [14]. For the relation of Wilson loops and large R-charge operators, see for example [10]. Also, the paper [12] has recently studied the anomalous dimensions on the field theory side.

In this paper, we would like to build on this success by applying the methods of [1] to gauge-gravity dualities that are much less understood than the canonical $AdS_5 \times S^5$ string theory with $\mathcal{N} = 4$ super Yang-Mills (SYM) [19, 20, 21, 22] and the immediate derivatives thereof. One would like to understand dual descriptions to $\mathcal{N} = 1$ gauge theories in four dimensions, which have more in common with observed particle physics. M-theory on a non-compact G_2 holonomy manifold is one way of realising such a gravity dual, as we shall now summarise.

1.2 Background to the G_2 holonomy duality

Progress in this direction originated from the duality between Chern-Simons gauge theory on S^3 at large N and topological string theory on a blown up Calabai-Yau conifold [23]. This duality was embedded in string theory as a duality between the IIA string theory of N D6-branes wrapping the blown up S^3 of the deformed conifold and IIA string theory on the small resolution of the conifold with N units of two form Ramond-Ramond flux through the blown up S^2 and no branes [24]. The D6-brane side of the duality involves an $\mathcal{N} = 1$ gauge theory in four dimensions that is living on the non-compact directions of the branes, at energies that do not probe the wrapped S^3.

Before lifting this duality to M-theory, let us make a few further statements regarding the relation of the D6 branes to the field theory. In order for the wrapped branes to preserve some supersymmetry, one has to embed the spin connection of the wrapped cycle into the gauge connection, which is known as twisting the theory. On the wrapped part of the brane, the gauge theory is topological [25]. Whilst the twisting allows the configuration to preserve supersymmetry, some of the supercharges will not have massless modes. Therefore the theory living on the flat part of the brane will preserve a lower fraction of supersymmetry than the unwrapped flat brane configuration.

When we have flat D6 branes, the symmetry group of the configuration is $SO(1,6) \times SO(3)_R$. The spinors transform in the **(8,2)** of the isometry group and the scalars in the **(1,3)**, whilst the gauge particles are in the **(6,1)** [26]. Wrapping the D6 brane on the three-sphere breaks the group to $SO(1,3) \times SO(3) \times SO(3)_R$. The technical meaning of twisting is that the two SO(3)s get mixed to allow the existence of four dimensional spinors that transform as scalars under the new twisted SO(3). One can then see that the remaining particles in the spectrum that transform as scalars under the twisted SO(3) are the gauge field and four of the initial sixteen spinors. Thus the field content is that of $\mathcal{N} = 1$ SYM. Apart from these fields, there will be massive modes, whose mass scale is set by the size of the curved cycle. When we probe the system with low enough energies, we find only the spectrum of $\mathcal{N} = 1$ SYM. In the following when we consider 'high energies', we will be understanding that the energies are not high enough to probe the massive modes of the theory.

The situation is not quite as straightforward as outlined above. This is because for D6 branes in flat space, the 'decoupling' limit does not completely decouple the gauge theory modes from bulk modes [27]. In our case, we expect a good gauge theory description only when the size of the wrapped three-cycle is large, which implies that we have to probe the system with very low energies to get 3+1 dimensional SYM [28]. In this case, the size of the two cycle in the flopped geometry is very near to zero, so a good gravity description is not expected. In short, we must keep in mind that the field theory we will be dealing with has more degrees of freedom than pure $\mathcal{N} = 1$ SYM.

It was discovered that the duality described above is naturally understood by considering M-theory on a G_2 holonomy metric [28]. In eleven dimensions, G_2 holonomy implements $\mathcal{N} = 1$ as pure gravity. One starts with a singular G_2 manifold that on dimensional reduction to IIA string theory corresponds to N D6 branes wrapping the S^3 of the deformed conifold. There is an SU(N) gauge theory at the singular locus/D6 brane. This configuration describes the UV of the gauge theory. As the coupling runs to the IR, a blown up S^3 in the G_2 manifold shrinks and another grows. This flop is smooth in M-theory physics. The metrics will be discussed in more detail in the following sections. In the IR regime, the G_2 manifold is non-singular and dimensional reduction to IIA gives precisely the aforementioned small resolution of the conifold with no branes and RR flux. Confinement emerges nicely in this picture, because the gauge degrees of freedom have disappeared in the IR along with the branes. The smooth M-theory physics of this process was systematised in [29] where it was shown that the transitions are in fact between three possible geometries, corresponding to the deformed conifold and two small resolutions of the conifold. This should be understood as a quantum mechanically broken triality symmetry. See also [30]. The M-theory lift of the IIA duality of [24] was arrived at independently in [31] from the perspective of studying the M-theory geometry describing four dimensional gauge theory localised at ADE singularities [32].

The moral of these discoveries would seem to be that special holonomy in eleven dimensions is a natural way to formulate the dual geometry of gauge theories living on wrapped D-branes. This approach was further pursued in, for example, [33, 34, 35]. The G_2 metrics describe the near horizons of branes as opposed to the full brane supergravity solution because they are not asymptotically flat. We cannot generically take a further near horizon limit of the metric, $r \to 0$ typically, because this would spoil the special holonomy and therefore the matching of supersymmeties. Working within this paradigm, we shall consider rotating membranes on eleven dimensional backgrounds $\mathbb{R}^{1,3} \times X_7$, where X_7 is a cohomogeneity-one non-compact G_2 manifold.

1.3 Motivation and contents of this work

To get going, we will first study rotating membrane configurations on $AdS_4 \times S^7$ and then go on to study rotating membranes on G_2 holonomy manifolds. The first step is something of a warm-up to show that one obtains sensible results by considering rotating membranes in a configuration that is fairly well studied. It is however, severely limited by the fact that comparatively little is known about the dual theory. The second step, on the other hand, is particularly interesting as the duality takes us to pure $\mathcal{N} = 1$ SYM theories. This is a theory that is understood and not so different from the gauge theories of nature. However, what is very poorly understood indeed is the precise nature of the duality with M-theory on G_2 holonomy spacetimes. The anomalous dimensions of operators with large quantum numbers exhibit very characteristic behaviours that seem be captured by fairly simple string/M-theory configurations. It thus provides a window into the duality.

Some rotating membrane configurations on $AdS_7 \times S^4$ were discussed in [13, 16]. We will show how a simple modification of their configurations will give logarithms in the energy-spin relation. This modification will later provide the inspiration for finding logarithms in the G_2 holonomy cases. Another previous use of membrane configurations in $AdS_7 \times S^4$ was in providing dual descriptions of Wilson loops in [36]. Also, the presently known matchings of $N = 1$ SYM with G_2 holonomy M-theory come from considering membrane instantons as gauge theory instantons that generate the superpotential [32], membranes wrapped on one-cycles in the IR geometry that are super QCD strings in the gauge theory [31, 37], and fivebranes wrapped on three-cycles that give domain walls in the gauge theory [31, 38]. These matchings are essentially topological and do not use the explicit form of the G_2 metrics. In this sense our results, which do use the explicit form of various metrics, are of a different nature from previous studies of the duality.

In section 2 we recall the basic formulae for supermembranes and fix notation. In section 3 we study rotating membranes on AdS spaces that are dual to gauge theories in three and four dimensions with varying amounts of supersymmetry. In particular we obtain various configurations with logarithmic anomalous dimensions. In section 4 we recall the existence of Asymptotically Localy Conical (ALC) G_2 and their role in the $\mathcal{N} = 1$ duality. We go on to study membranes rotating in these backgrounds. Again we obtain logarithmic anomalous dimensions, as well as a variety of other behaviours. In obtaining the logarithms we consider energy and charge densities of a non-compact brane. Section 5 contains a summary and discussion, a few comments regarding the dual operators to the membrane configurations, and open questions. The appendix A is independent from the rest of this work and sets up a formalism for studying strings moving on warped backgrounds. The appendix B explicitly checks the lack of supersymmetry of the G_2 holonomy configurations.

2. Membrane formulae

In this section, we briefly summarise the action, equations of motion and gauge fixing constraints for membranes. The bosonic sector of the supermembrane action [39] is

$$\mathbf{I}_B = \frac{-1}{(2\pi)^2 l_{11}^3}\int d^3\sigma \left(\frac{(-\gamma)^{1/2}}{2}\left[\gamma^{ij}\frac{\partial X^\mu}{\partial\sigma^i}\frac{\partial X^\nu}{\partial\sigma^j}G_{\mu\nu}(X) - 1\right] + \epsilon^{ijk}\frac{\partial X^\mu}{\partial\sigma^i}\frac{\partial X^\nu}{\partial\sigma^j}\frac{\partial X^\rho}{\partial\sigma^k}C_{\mu\nu\rho}(X)\right), \tag{2.1}$$

where $i,j,k = 0,\ldots,2$ and $\mu,\nu,\rho = 0,\ldots,10$. The worldsheet metric is γ_{ij}, the embedding fields are X^μ and the eleven dimensional background is described by the spacetime metric $G_{\mu\nu}$ and three-form field $C_{\mu\nu\rho}$. The corresponding field strength is $H = dC$.

The equations of motion are

$$\begin{aligned} \gamma_{ij} &= \partial_i X^\mu \partial_j X^\nu G_{\mu\nu}(X)\,, \\ \partial_i\left((-\gamma)^{1/2}\gamma^{ij}\partial_j X^\rho\right) &= -(-\gamma)^{1/2}\gamma^{ij}\partial_i X^\mu\partial_j X^\nu \Gamma^\rho_{\mu\nu}(X) - \\ &\quad -\epsilon^{ijk}\partial_i X^\mu \partial_j X^\nu \partial_k X^\sigma H^\rho_{\mu\nu\sigma}(X)\,. \end{aligned} \tag{2.2}$$

The action (2.1) is equivalent on shell to the Dirac-Nambu-Goto action. The three diffeomorphism symmetries of the action may be gauge fixed by imposing the following constraints

$$\begin{aligned} \gamma_{0\alpha} = \partial_0 X^\mu \partial_\alpha X^\nu G_{\mu\nu}(X) &= 0\,, \\ \gamma_{00} + L^2 \det[\gamma_{\alpha\beta}] = \partial_0 X^\mu\partial_0 X^\nu G_{\mu\nu}(X) + L^2\det[\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X)] &= 0\,, \end{aligned} \tag{2.3}$$

where $\alpha,\beta = 1,\ldots,2$ are the spatial worldsheet indices and L^2 is an arbitrary constant to be fixed later. Using the equation of motion for γ_{ij} (2.2) and the gauge fixing conditions (2.3), one obtains the action

$$\begin{aligned} \mathbf{I}_B = \frac{1}{2(2\pi)^2 L l_{11}^3}\int d^3\sigma \Big(&\partial_0 X^\mu \partial_0 X^\nu G_{\mu\nu}(X) - L^2 \det[\partial_\alpha X^\mu\partial_\beta X^\nu G_{\mu\nu}(X)] + \\ &+ 2L\epsilon^{ijk}\partial_i X^\mu \partial_j X^\nu \partial_k X^\rho C_{\mu\nu\rho}(X)\Big). \end{aligned} \tag{2.4}$$

Note that for backgrounds where the C field does not couple to the membrane, the second constraint in (2.3) is just the constancy of the hamiltonian of the action (2.4).

For the simple configurations we consider below, that have additional conserved charges, the equations of motion will almost follow from imposing the constant hamiltonian constraint. However, in making ansätze for solutions one needs to be quite careful about consistency. Warped terms are particularly dangerous. The equations of motion for the gauge fixed action (2.4) contain terms like $\partial_0 X^\mu \partial_0 X^\nu \partial_\rho G_{\mu\nu}(X)$, and these generally have to vanish in order for the equations of motion to be solved. This must be checked for each anstaz adopted.

3. Membranes rotating in AdS geometries

This section considers membranes rotating in various AdS backgrounds. These configurations are very straightforward generalisations of previous work and we consider this section to be a warm-up for the G_2 cases to be considered below. We modify previous configurations slightly to obtain logarithmic terms in energy-spin relations. We call these new configurations type I and the previously studied, non-logarithmic, membrane configurations type II. We emphasise that this distinction, and the existence of logarithms, is independent of the precise AdS geometry, so long as the internal manifold has a U(1) isometry.

3.1 Membranes rotating in $AdS_4 \times M_7$

We start by studying membranes moving in $AdS_4 \times M_7$. We will take first the maximally supersymmetric case with $M_7 = S^7$ and then move on to more interesting geometries preserving $\mathcal{N} = 1, 2, 3$ supersymmetries in the dual 2+1 dimensional theory. The dual field theories will be conformal and are, in some aspects, very well known. We will study two different types of configurations. The first type of configurations, type I, are similar to the original string configurations [1], and will give logarithmic anomalous dimensions. Type II configurations are essentially the membrane configurations that have already been studied [13].

The metric and three-form potential are

$$\begin{aligned}\frac{1}{l_{11}^2} ds_{11}^2 &= -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho(d\alpha^2 + \sin^2\alpha d\beta^2) + B^2 ds_7^2\,, && (3.1)\\ H &= k\cosh\rho\sinh^2\rho\sin\alpha dt\wedge d\rho\wedge d\alpha\wedge d\beta\,, && (3.2)\\ C &= -\frac{k\sin\alpha}{3}\sinh^3\rho dt\wedge d\alpha\wedge d\beta. && (3.3)\end{aligned}$$

Here B is the relative radius of AdS_4 with respect to the seven-manifold, whilst k is a number that can be easily determined from the equations of motion.

Let us first study the case in which $M_7 = S^7$. In this case we find it convenient to write the metric as

$$\begin{aligned}ds_7^2 = 4d\xi^2 &+ \cos^2\xi(d\theta^2 + d\phi^2 + d\psi^2 + 2\cos\theta d\phi d\psi) + \\ &+ \sin^2\xi(d\tilde{\theta}^2 + d\tilde{\phi}^2 + d\tilde{\psi} + 2\cos\tilde{\theta}d\tilde{\phi}d\tilde{\psi})\,. \qquad (3.4)\end{aligned}$$

We could equally well be considering the case of the squashed seven sphere, $\tilde{S}^7$, the supergravity system will be dual to a conformal gauge theory with $\mathcal{N} = 1$ supersymmetry. In this case, the metric will read

$$ds_7^2 = d\Omega_4^2 + \frac{1}{5}(\omega^i - A^i)^2\,, \qquad (3.5)$$

with A^i being the SU(2) one-instanton on S^4 and ω^i the left-invariant one-forms of SU(2) (for details see for example [40]). We will see that we obtain the same results in both cases.

The two types of solution mentioned above differ in the dependence of the AdS coordinates on the worldvolume of the membrane. The membrane is moving forward trivially in time, one direction is stretched along the radial direction of AdS and is rotating either in the AdS (spin) or in the internal, M_7, space (R-charge). There is one extra direction left, with worldvolume coordinate δ, that distinguishes the membrane from a string. We must wrap this direction along a U(1) isometry. This could either be in the AdS (type-I configuration), or in the internal space (type-II configuration). Thus, for the type-I solutions the wrapped direction of the membrane, δ-direction, remains finite at infinity, and the long membrane limit is string-like. For the type-II solutions, the wrapped direction is not stabilised asymptotically. This kind of distinction will play an important role below when we discuss membranes on G_2 manifolds.

In constracting these solutions, it is important to check that the ansätze are in fact consistent. In practice, this constrains the values that one may give to constant angular coordinates.

3.2 Solutions of type I: logarithms

As this is our first configuration, let us describe it clearly. We want to embed the membrane into spacetime such that it is moving trivially forward in time and is extended along the radial direction of AdS

$$t = \kappa\tau\,, \qquad \rho = \rho(\sigma)\,. \tag{3.6}$$

We would then like to have the membrane rotating in the AdS space

$$\beta = \omega\tau\,, \qquad \alpha = \frac{\pi}{2}\,, \tag{3.7}$$

and for consistency of the anstaz it turns out that we cannot have the membrane rotating in the internal sphere at the same time, so the configuration will not have any R-charge,

$$\psi = 0\,, \qquad \theta = \xi = \frac{\pi}{2}\,, \qquad \phi = 0\,. \tag{3.8}$$

Finally, we wrap the membrane along a U(1) in the sphere

$$\tilde{\psi} = \lambda\delta\,, \qquad \tilde{\theta} = \frac{\pi}{2}\,, \qquad \tilde{\phi} = 0\,. \tag{3.9}$$

Note that in (3.1), the size of the M_7 space does not change with the AdS radial direction ρ and therefore the wrapped direction remains stabilised at infinity. This will be the main difference with the type-II solutions below.

We can check that two of the constraints (2.3) are satisfied

$$G_{\mu\nu}\partial_\sigma X^\mu \partial_\tau X^\nu = G_{\mu\nu}\partial_\delta X^\mu \partial_\tau X^\nu = 0\,, \tag{3.10}$$

whilst the remaining constraint

$$\frac{1}{L^2} G_{\mu\nu}\partial_\tau X^\mu \partial_\tau X^\nu = (G_{\mu\nu}\partial_\sigma X^\mu \partial_\delta X^\nu)^2 - (G_{\mu\nu}\partial_\sigma X^\mu \partial_\sigma X^\nu)(G_{\mu\nu}\partial_\delta X^\mu \partial_\delta X^\nu)\,, \tag{3.11}$$

gives the following relation, upon choosing $L = 1/\lambda$,

$$\left(\frac{d\rho}{d\sigma}\right)^2 = \frac{1}{l_{11}^2 B^2}\left[\kappa^2\cosh^2\rho - \omega^2\sinh^2\rho\right]. \tag{3.12}$$

We may now compute the action by substituting into the formulae of section 2,

$$I = -P \int d\tau \int_0^{\rho_0} d\rho \sqrt{\kappa^2 \cosh^2 \rho - \omega^2 \sinh^2 \rho}\ , \tag{3.13}$$

where $P = 16\pi|B|/(2\pi)^2$ is a normalization factor and ρ_0 is the turning point given by

$$\left.\frac{d\rho}{d\sigma}\right|_{\rho_0} = 0 \Leftrightarrow \tanh \rho_0 = \frac{\kappa}{\omega}\,. \tag{3.14}$$

Note that the term in the action associated to the three form vanishes. There is a factor of four in the normalisation because of the periodicity of the integrand and the fact that the membrane doubles back on itself. Write the integrals defining the conserved energy and spin by differentiating the lagrangian

$$E = -\frac{\delta I}{\delta\kappa} = P\kappa \int_0^{\rho_0} d\rho \frac{\cosh^2 \rho}{\sqrt{\kappa^2 \cosh^2 \rho - \omega^2 \sinh^2 \rho}}\,, \tag{3.15}$$

$$S = \frac{\delta I}{\delta\omega} = P\omega \int_0^{\rho_0} d\rho \frac{\sinh^2 \rho}{\sqrt{\kappa^2 \cosh^2 \rho - \omega^2 \sinh^2 \rho}}\,, \tag{3.16}$$

Now one needs to do the integrals. Fortunately, these integrals are exactly the ones considered for rotating strings [1], and therefore we may just read off the results from these papers. What one is interested in is the relationship between the spin and energy for large and small energy. In particular, for long membranes we will get the result

$$E - S = P \ln \frac{S}{P} + \cdots\,. \tag{3.17}$$

To do the integrals one uses the endpoint constraint (3.14) and also one sometimes needs to use the normalisation condition

$$2\pi = \int d\sigma = \int_0^{r_0} d\rho \frac{d\sigma}{d\rho}\,. \tag{3.18}$$

It is perhaps not surprising that a membrane wrapped on a cycle of constant size has the same behaviour as a string.

3.3 Solutions of type II

We consider now a configuration that is similar to the configuration of the previous subsection, but with the difference that the wrapped direction is in the AdS space and not the sphere. That is

$$\beta = \lambda\delta\,, \qquad \alpha = \frac{\pi}{2}\,, \tag{3.19}$$

compare this with (3.7) and (3.9). The rotation must now be in the sphere only, as there are no more directions in the AdS

$$\tilde{\psi} = \nu\tau\,, \qquad \tilde{\theta} = \xi = \frac{\pi}{2}\,, \qquad \tilde{\phi} = 0\,. \tag{3.20}$$

The remaining directions are then

$$t = \kappa\tau\,, \qquad \rho = \rho(\sigma)\,, \qquad \theta = \frac{\pi}{2}\,, \qquad \phi = 0\,, \qquad \psi = 0\,. \tag{3.21}$$

The first two constraints are satisfied as before, whilst (3.11) gives

$$\left(\frac{d\rho}{d\sigma}\right)^2 = \frac{1}{l_{11}^2 \sinh^2\rho}\left[\kappa^2\cosh^2\rho - B^2\nu^2\right]. \tag{3.22}$$

The action will be

$$I = -P\int d\tau \int_0^{\rho_0} d\rho \sinh\rho\sqrt{\kappa^2\cosh^2\rho - B^2\nu^2}\,, \tag{3.23}$$

the limits of integration are zero and ρ_0 is the solution of the endpoint constraint, which is now $\cosh^2\rho_0 = B^2\nu^2/\kappa^2$. The normalization constant is $P = 8\pi|B|/(2\pi)^2$ and the contribution of the $C^{(3)}$ field vanishes as before.

One can now write down the integrals defining the energy and R-charge angular momentum, there is no room for spin in this case due to the fact that we are dealing with AdS_4,

$$E = -\frac{\delta I}{\delta\kappa} = P\kappa\int_0^{\rho_0} d\rho\frac{\sinh\rho\cosh^2\rho}{\sqrt{\kappa^2\cosh^2\rho - B^2\nu^2}}, \tag{3.24}$$

$$J = \frac{\delta I}{\delta\nu} = PB^2\nu\int_0^{\rho_0} d\rho\frac{\sinh\rho}{\sqrt{\kappa^2\cosh^2\rho - B^2\nu^2}}\,. \tag{3.25}$$

Again, we now recognise these integrals from previous work, this time from rotating membranes [13]. Thus we may again just read off the energy-R-charge relations. For long membranes, these are of the form $E = J + \cdots$.

A type-II configuration for membranes in $AdS_7 \times S^4$ has been discussed in [13]. Clearly, one can also write down a type-I configuration in $AdS_7 \times S^4$ and obtain a logarithmic $E - S$ in that background.

3.4 The case of $AdS_4 \times Q^{1,1,1}$

We consider now the case where the internal manifold M_7 of equation (3.1) is $Q^{1,1,1}$. This manifold is a U(1) fibration over $S^2 \times S^2 \times S^2$. The interest of this configuration is that it provides an M theory dual to a three dimensional $\mathcal{N} = 2$ conformal field theory. This is an interesting field theory, that can be thought of as describing low energy excitations living on M2 branes, that are placed on the tip of an eight dimensional cone with special holonomy. The theory is described in terms of fields A_i, B_i, C_i with $i = 1, 2$ and with given transformation properties under the colour and flavour groups. Gauge invariant operators are of the form $X = ABC$ and can be put in correspondence with supergravity modes in AdS_4. Besides, baryonic operators can be constructed. This theory was well studied in various papers [41, 42, 40].

The eleven dimensional configuration reads

$$\begin{aligned}\frac{1}{l_{11}^2}ds_{11}^2 &= -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho(d\alpha^2 + \sin^2\alpha d\beta^2) + \\ &\quad + \frac{1}{8}(d\Omega_2(\theta_1,\phi_1) + d\Omega_2(\theta_2,\phi_2) + d\Omega_2(\theta_3,\phi_3)) + \\ &\quad + \frac{1}{16}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2 + \cos\theta_3 d\phi_3)^2 \,, \qquad (3.26)\\ H &= k\cosh\rho\sinh^2\rho\sin\alpha dt\wedge d\rho\wedge d\alpha\wedge d\beta \,, \qquad (3.27)\\ C &= -\frac{k\sin\alpha}{3}\sinh^3\rho dt\wedge d\alpha\wedge d\beta \qquad (3.28)\end{aligned}$$

Here again, k is a constant determined by the equations of motion.

We can again consider two types of solutions. We will be brief in this case, since the calculations results are not very different from those of the previous subsections. Indeed the main point here is that the existence of two types of energy-spin relations, one with logarithms and one without, is independent of the internal manifold, so long as it has a U(1) isometry around which we can wrap the membrane.

Type I solutions. In this case the configuration reads reads

$$t = \kappa\tau\,, \quad \rho = \rho(\sigma)\,, \quad \beta = \omega\tau\,, \quad \alpha = \theta_i = \frac{\pi}{2}\,, \quad \phi_i = \nu_i\tau\,, \quad \psi = \lambda\delta\,. \qquad (3.29)$$

The constraint reads

$$\left(\frac{d\rho}{d\sigma}\right)^2 = \frac{16}{l_{11}^2}\left(-\omega^2\sinh^2\rho + \kappa^2\cosh^2\rho - \frac{1}{8}(\nu_1^2+\nu_2^2+\nu_3^2)\right), \qquad (3.30)$$

there is a turning point where $d\rho/d\sigma = 0$ and the action reads

$$I = -P\int d\tau\int_0^{\rho_0} d\rho\frac{1}{4}\sqrt{\kappa^2\cosh^2\rho - \omega^2\sinh^2\rho - \frac{1}{8}(\nu_1^2+\nu_2^2+\nu_3^2)}\,. \qquad (3.31)$$

In this case, the normalization factor is $P = 16\pi/(2\pi)^2$. As in the previous sections, the contribution of the $C^{(3)}$ field vanishes. We can compute the energy, spin and R-charge angular momentum, and the results are essentially identical to those coming from equations (3.15)–(3.18). In particular, there is a logarithmic $E - S$ relation.

Type II solutions. In this case the solution will read

$$t = \kappa\tau\,, \quad \rho = \rho(\sigma)\,, \quad \beta = \lambda\delta\,, \quad \alpha = \theta_i = \frac{\pi}{2}\,, \quad \psi = 0\,, \quad \phi_i = \nu_i\tau\,. \qquad (3.32)$$

The constraint will give a turning point ρ_0, when $d\rho/d\sigma = 0$

$$\left(\frac{d\rho}{d\sigma}\right)^2 l_{11}^2\sinh^2\rho = \kappa^2\cosh^2\rho - \frac{1}{8}(\nu_1^2+\nu_2^2+\nu_3^2)\,, \qquad (3.33)$$

and the action will be

$$I = P\int d\tau\int_0^{\rho_0} d\rho\sinh\rho\sqrt{\kappa^2\cosh^2\rho - \frac{1}{8}(\nu_1^2+\nu_2^2+\nu_3^2)}\,. \qquad (3.34)$$

The normalization factor will be $P = 8\pi/(2\pi)^2$. This time, the results will be essentially the same as those coming from the previous type-II configuration, of equations (3.24)–(3.25).

It is not difficult to see from (3.5) that one may obtain the same results using the squashed seven sphere, as we have the same cycles on which to wrap the membrane and rotate. Thus we obtain membrane configurations dual to operators of an $\mathcal{N}=1$ theory in three dimensions.

We can also consider the case of three dimensional $\mathcal{N}=3$ conformal field theories. These theories are dual to geometries of the form $AdS_4 \times N^{0,1,0}$, where the manifold $N^{0,1,0}$ has metric

$$ds_7^2 = d\xi^2 + \frac{1}{4}\sin^2\xi(\sigma_1^2+\sigma_2^2+\cos^2\xi\sigma_3^2) + \frac{1}{2}(\omega^i - A^i)^2 \tag{3.35}$$

with $A^{1,(2)} = \cos\xi\sigma_{1,(2)}$, $2A^3 = (1+\cos^2\xi)\sigma_3$, and ω^i, σ_i are left-invariant forms in the different SU(2)s. This type of field theory is interesting because it has the same field content as $\mathcal{N}=4$ theories, but there are fermionic interactions that only preserve $\mathcal{N}=3$. Following the steps above, one can find type-I and type-II solutions for these metrics. Everything will work as before, with different numerical coefficients.

It should be clear by now that all that is needed to obtain a logarithmic configuration is a stabilised circle to wrap the membrane on. As we will see below in the section on G_2 manifolds, this does seems work in more general situations than AdS product spacetimes.

3.5 Membranes moving in warped $AdS_5 \times M_6$ spaces

We now consider membranes moving in a geometry that is dual to an $\mathcal{N}=2$ supersymmetric conformal field theory in four dimensions, as opposed to the three dimensions of the previous cases. The eleven dimensional configuration was written in [43] and represents M5 branes wrapping a hyperbolic two-manifold. The geometry has the form of a warped product of five dimensional AdS space times a six dimensional manifold. This should be thought of as M5 branes wrapping some compact (hyperbolic) cycle inside a Calabi-Yau two-fold.

The metric looks, schematically (for a detailed discussion see [44]), as follows

$$\begin{aligned}\frac{1}{l_{11}^2}ds^2 &= \Delta(\theta)^{1/3}\left(R_a^2 ds_{AdS_5}^2 + R_b^2(d\tilde{\theta}^2 + \sinh^2\tilde{\theta}d\tilde{\phi}^2) + R_c^2 d\theta^2\right) + \\ &\quad + R_c^2\frac{\Delta(\theta)^{-2/3}}{4}\left(\cos^2\theta(d\alpha^2+\sin^2\alpha d\beta^2) + 2\sin^2\theta(d\psi - \cosh\tilde{\theta}d\tilde{\phi})^2\right),\end{aligned} \tag{3.36}$$

where AdS_5 is written in the coordinates $(\rho, t, \xi_1, \xi_2, \phi)$ as usual, $\Delta(\theta) = 1 + \cos^2\theta$ and the R_i are constants. The $C^{(3)}$ field has the schematic form

$$C = f_1(\alpha,\theta,\tilde{\theta})d\tilde{\theta}\wedge d\tilde{\phi}\wedge d\beta + f_2(\alpha,\theta)d\theta\wedge d\beta\wedge(d\psi - \cosh\tilde{\theta}d\tilde{\phi})\,. \tag{3.37}$$

We can consider a solution of the form

$$\rho = \rho(\sigma)\,, \quad t = \kappa\tau\,, \quad \phi = \omega\tau\,, \quad \psi = \lambda\delta\,, \quad \theta = \frac{\pi}{2}\,. \tag{3.38}$$

The remaining angles take values of 0 or $\pi/2$ as in previous configurations. Note that $\theta = \pi/2$ is necessary to solve the equations of motion. There is no R-charge in this

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configuration. Warped solutions are discussed in more detail in appendix A below. The key point about this configuration is that the warping factor is unimportant because we fix a value of θ so that it just becomes an overall number. The configuration is of type I because the wrapped direction, $\psi = \lambda\delta$, is in the M_6, therefore we expect to get relations of the form $E - S \sim \ln S$, and indeed this is what one finds upon doing the calculations. The integrals that emerge are, up to numerical coefficients, the same as the type-I configurations we studied above.

3.6 Membranes moving near an AdS black hole

For completeness, we briefly consider now the case of membranes orbiting in an eleven dimensional geometry given by

$$\frac{1}{l_{11}^2} ds_{11}^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\alpha^2 + \sin^2\alpha d\beta^2) + B^2 ds_{S^7}^2 \,. \tag{3.39}$$

With the function

$$f(r) = 1 + \frac{r^2}{R^2} - \frac{M^2}{r} \,, \tag{3.40}$$

and $B = R$, the previous metric is a black hole in AdS_4. A very nice physical description of the AdS/CFT correspondence for strings orbiting about black holes was given in [7]. We shall limit ourselves to discussing the membrane configurations and energy spin relations. As previously, we will consider two types of configuration, the type I

$$t = \kappa\tau \,, \quad \rho = \rho(\sigma) \,, \quad \beta = \omega\tau \,, \quad \alpha = \theta_i = \tilde{\theta}_i = \frac{\pi}{2} \,, \quad \tilde{\psi} = \lambda\delta \,, \quad \xi = \frac{\pi}{4} \,, \tag{3.41}$$

and the type II

$$t = \kappa\tau \,, \quad \rho = \rho(\sigma) \,, \quad \beta = \lambda\delta \,, \quad \alpha = \theta_i = \tilde{\theta}_i = \frac{\pi}{2} \,, \quad \tilde{\psi} = \nu\tau \,. \tag{3.42}$$

We can construct the expression for the membrane constraint in the first case,

$$2f(r)(-\omega^2 r^2 + \kappa^2 f(r)) = l_{11}^2 B^2 \left(\frac{dr}{d\sigma}\right)^2 \,, \tag{3.43}$$

and for the type-II solutions

$$f(r)(-\nu^2 B^2 + 2\kappa^2 f(r)) = 2r^2 l_{11}^2 \left(\frac{dr}{d\sigma}\right)^2 \,. \tag{3.44}$$

Upon requiring $dr/d\sigma$ to vanish at the endpoints, we will obtain two different values of $r_{\min}, r_{\max}$, that is, the integration limits in the action, when we change variables from σ to the radial coordinate r. This is physically the fact that the membrane is entirely outside the event horizon and is therefore orbiting rather than rotating.

The expression for the action, energy and spin, in the type I case is,

$$I = P \int d\tau \int_{r_{\min}}^{r_{\max}} \sqrt{\kappa^2 f(r) - \omega^2 r^2} \frac{dr}{\sqrt{f(r)}} \,, \tag{3.45}$$

$$E = \kappa P \int_{r_{\min}}^{r_{\max}} \frac{\sqrt{f(r)}}{\sqrt{\kappa^2 f(r) - \omega^2 r^2}} dr \,, \tag{3.46}$$

$$S = \omega P \int_{r_{\min}}^{r_{\max}} \frac{r^2}{\sqrt{\kappa^2 f^2(r) - \omega^2 r^2 f(r)}} dr \,, \tag{3.47}$$

while for the type-II configurations we will have for the action, energy and R-symmetry angular momentum

$$I = \frac{P}{2}\int d\tau \int_{r_{\min}}^{r_{\max}} \sqrt{\kappa^2 f(r) - \nu^2 B^2/2}\frac{r}{\sqrt{f(r)}}dr \,, \tag{3.48}$$

$$E = \frac{\kappa P}{2}\int_{r_{\min}}^{r_{\max}} \frac{r\sqrt{f(r)}}{\sqrt{\kappa^2 f(r) - \nu^2 B^2/2}}dr \,, \tag{3.49}$$

$$J = \frac{\nu P B^2}{4}\int_{r_{\min}}^{r_{\max}} \frac{r}{\sqrt{\kappa^2 f^2(r) - \nu^2 B^2 f(r)/2}}dr \,. \tag{3.50}$$

Let us study the explicit expressions for the energy and the spin of type I configurations. It is convenient to make a choice of parameters $M = R = \kappa = 1$. The results will remain true at least for a small interval of value for M around $M = 1$. We can see that for values of the parameter ω close to one, corresponding to long membranes, the functions inside the square roots are positive in the interval r_h, r_+, where r_h is the root of $r^3 + r - 1$ and the roots of $-(\omega^2 - 1)r^3 + r - 1$ are r_-, r_+ both of them positive and bigger than r_h and r_*, a third negative root. The integrals read,

$$E \sim \int_{r_-}^{r_+} dr \sqrt{\frac{r^3 + r - 1}{-(\omega^2 - 1)r^3 + r - 1}} \,, \tag{3.51}$$

$$S \sim \int_{r_-}^{r_+} dr \frac{r^3}{\sqrt{[r^3 + r - 1][-(\omega^2 - 1)r^3 + r - 1]}} \,. \tag{3.52}$$

Now, we want to study the approximate expressions for these integrals in the cases of long membranes, that is membranes that are extended in the interval (r_-, r_+). Evaluating the approximate expressions for the integrals in the case of long membranes, we get a relation of the form $E - kS \sim S^3$.

4. Rotating membranes on G_2 manifolds

4.1 The duality with ALC G_2 metrics

Partially motivated by the developments described in the introduction, there has been significant recent progress in constructing new cohomogeneity-one manifolds with G_2 holonomy [45, 30, 46, 47, 48, 49, 50, 51], generalising the manifolds that have been known for some time [52, 53].

When the M-theory flop was discussed in [28], the only known G_2 metric with the necessary symmetries to describe wrapped D6 branes in type IIA was asymptotically a cone over $S^3 \times S^3$ [52, 53]. The essential point is that one S^3 collapses at the origin whilst another does not. [1] Thus depending on which S^3 the M-theory cycle is contained in, one gets either a IIA reduction that is singular at the origin — branes — or a non-singular reduction — no branes. However, in these metrics the dilaton diverges at infinity after

[1] Note that the collapsing and non-collapsing S^3s need not coincide with the two S^3s in terms of which the metric is written.

reduction so they are unsatisfactory IIA backgrounds. The authors of [28] thus postulated the existence of two new types of G_2 holonomy metric to fix this problem. These metrics should not be Asymptotically Conical (AC), but Asymptotically Locally Conical (ALC), that is to say that at infinity there should be a circle with a stabilised radius. This circle will be the M-theory circle and the corresponding IIA dilaton will be well-behaved. The two metrics would correspond to when the stabilised U(1) is contained within the collapsing S^3 or the non-collapsing S^3, corresponding to good D6-brane or good non-D6-brane solutions, respectively.

This picture was essentially realised with the discovery of explicit ALC G_2 metrics. G_2 metrics reducing to D6-branes wrapping the deformed conifold were discussed in [45, 48, 46], these are called the $\mathbb{B}_7$ family of metrics. Metrics reducing to the small resolution of the conifold with fluxes were discussed in [49, 51, 50], these are called the $\mathbb{D}_7$ family. Transformations of these metrics under the broken triality symmetry were discussed in [51], this does not change the radial behaviour or the symmetries.

The situation is not quite as anticipated by [28]. All the known G_2 metrics are constructed out of left-invariant one-forms on the two $S^3 = \mathrm{SU}(2)$s, $\{\Sigma_i, \sigma_i\}_{i=1}^3$, see e.g. (4.2) below. In the AC case, there is an $\mathrm{SU}(2)^\Sigma_L \times \mathrm{SU}(2)^\sigma_L \times \mathrm{SU}(2)^D_R$ isometry group, where the $\mathrm{SU}(2)^\Sigma_L \times \mathrm{SU}(2)^\sigma_L$ part is manifest and corresponds to left multiplication on the spheres. The remaining $\mathrm{SU}(2)^D_R$ is a diagonally acting right multiplication. Note that right multiplication acts on the left-invariant forms as an adjoint action. The ALC metrics have a reduced isometry group $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, where an SU(2) has been broken to U(1) by the stabilised cycle. It was suggested in [28] that the SU(2) that should be broken to U(1) would be $\mathrm{SU}(2)^\Sigma_L$ on one side of the flop and $\mathrm{SU}(2)^\sigma_L$ on the other side. This fits nicely with symmetry of their discussion. However, in order to realise this, one would need to construct G_2 metrics that were not written in terms of left invariant forms, as these automatically imply $\mathrm{SU}(2)^\Sigma_L \times \mathrm{SU}(2)^\sigma_L$ invariance. It is not clear how one would go about doing this. Instead, the solutions of the $\mathbb{B}_7$ and $\mathbb{D}_7$ families have the $\mathrm{SU}(2)^D_R$ broken to U(1). This is compatible with writing the metric in terms of the left invariant one forms.

There is a $\mathbb{Z}_N$ quotient of the metric that is responsible for the N D-branes or the N units of flux upon reduction to IIA. The $\mathbb{Z}_N$ always acts within the diagonal U(1) which is furthermore the M-theory circle. The action is singular at the origin in the $\mathbb{B}_7$ case, but not for $\mathbb{D}_7$.

It is important to observe that both the $\mathbb{B}_7$ and $\mathbb{D}_7$ families are two parameter families and one can go from one to the other [50], via the singular AC metric. Besides the scale parameter which measures the distance from the singular conical point, there is another parameter which measures the distance from the AC metric. The AC metric is contained in both families. Consider now the running of the coupling constant, described in section 1. Starting in a $\mathbb{B}_7$ metric in the UV, the flow must involve not only one of the S^3s shrinking — change of scale parameter — but also a flow towards the AC metric. This allows the flow to move to the $\mathbb{D}_7$ family via the AC metric as well as expanding a different S^3 in the IR. Thus the flow must move nontrivially in a two dimensional parameter space. A priori, it is not obvious why starting from an M-theory geometry that has a well behaved dilaton in the IIA reduction in the UV ($\mathbb{B}_7$ family), one should generically end up with an

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M-theory geometry that also admits a good IIA reduction in the IR ($\mathbb{D}_7$ family). But the desired flow should exist, which is enough to establish the IIA duality from M-theory with a well-behaved dilaton. Assuming that the quantum smoothing of the process continues to occur as it did in the AC case [29].

We will consider membranes rotating on all of the geometries discussed in this subsection. The $\mathbb{D}_7$ family are, strictly speaking, the gravity duals that describe the field theory in the IR. The precise role of the $\mathbb{B}_7$ metrics in the duality is unclear, although it could well be related to the lack of brane-bulk decoupling discussed in the introduction.

4.2 Membranes on the asymptotically conical metric and the $\mathbb{B}_7$ family

4.2.1 Metric formulae

The background is pure geometry, the three-form C-field is zero. The eleven dimensional metric is of the form

$$\frac{1}{l_{11}^2} ds_{11}^2 = -dt^2 + dx^2 + dy^2 + dz^2 + ds_7^2 \,, \tag{4.1}$$

where the G_2 metrics are

$$\begin{aligned} ds_7^2 = dr^2 &+ a(r)^2 \left[(\Sigma_1 - \sigma_1)^2 + (\Sigma_2 - \sigma_2)^2 \right] + d(r)^2 (\Sigma_3 - \sigma_3)^2 + \\ &+ b(r)^2 \left[(\Sigma_1 + \sigma_1)^2 + (\Sigma_2 + \sigma_2)^2 \right] + c(r)^2 (\Sigma_3 + \sigma_3)^2 \,, \end{aligned} \tag{4.2}$$

where Σ_i, σ_i are left-invariant one-forms on SU(2)

$$\begin{aligned} \sigma_1 &= \cos\psi_1 d\theta_1 + \sin\psi_1 \sin\theta_1 d\phi_1 \,, \\ \sigma_2 &= -\sin\psi_1 d\theta_1 + \cos\psi_1 \sin\theta_1 d\phi_1 \,, \\ \sigma_3 &= d\psi_1 + \cos\theta_1 d\phi_1 \,, \end{aligned} \tag{4.3}$$

where $0 \le \theta_1 \le \pi$, $0 \le \phi_1 \le 2\pi$, $0 \le \psi_1 \le 4\pi$, at least before including the $\mathbb{Z}_N$ quotient. The definitions for Σ_i are analogous but with $(\theta_1, \phi_1, \psi_1) \to (\theta_2, \phi_2, \psi_2)$. These metrics are locally asymptotic to cones over $S^3 \times S^3$. There is a finite size S^3 bolt at the origin. There is a two parameter family of such G_2 metrics, called $\mathbb{B}_7$ in the classification of [49, 50]. The radial functions satisfy the equations [45, 46]

$$\begin{aligned} \dot{a} &= \frac{1}{4}\left[-\frac{a^2}{bd} + \frac{d}{b} + \frac{b}{d} + \frac{c}{a} \right], & \dot{b} &= \frac{1}{4}\left[-\frac{b^2}{ad} + \frac{d}{a} + \frac{a}{d} - \frac{c}{b} \right], \\ \dot{d} &= \frac{1}{2}\left[-\frac{d^2}{ab} + \frac{a}{b} + \frac{b}{a} \right], & \dot{c} &= \frac{1}{4}\left[\frac{c^2}{b^2} - \frac{c^2}{a^2} \right]. \end{aligned} \tag{4.4}$$

Two exact solutions are known. One is the asymptotically conical (AC) solution of [52, 53], which has $\mathrm{SU}(2)^3 \times \mathbb{Z}_2$ symmetry. The other is only Asymptotically Locally Conical (ALC), with a stabilised U(1) at infinity [45, 48], which has $\mathrm{SU}(2)^2 \times \mathrm{U}(1) \times \mathbb{Z}_2$ symmetry. The remaining metrics in this family are only known numerically. Fortunately, we only require the asymptotics at the origin and at infinity, which are easily calculated from (4.4). As

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$r \to 0$ we have

$$
\begin{aligned}
a(r) &= R_0 + \frac{r^2}{16R_0} - \frac{(7+64q_0)r^4}{2560R_0^3} + \cdots\,,\\
b(r) &= \frac{r}{4} + \frac{q_0 r^3}{R_0^2} - \frac{(-1+98304q_0^2+1344q_0)r^5}{10240R_0^4} + \cdots\,,\\
c(r) &= \frac{r}{4} - \frac{(1+128q_0)r^3}{64R_0^2} + \frac{(216q_0+1+16896q_0^2)r^5}{640R_0^4} + \cdots\,,\\
d(r) &= R_0 + \frac{r^2}{16R_0} + \frac{(-3+64q_0)r^4}{1280R_0^3} + \cdots\,,
\end{aligned}
\tag{4.5}
$$

where q_0 and R_0 are constants. Note that $b(r)$ and $c(r)$ collapse, whilst $a(r)$ and $d(r)$ do not. As $r \to \infty$ we have

$$
\begin{aligned}
a(r) &= \frac{r}{2\sqrt{3}} + \frac{R_1}{2}(2q_1+\sqrt{3}) + \frac{3\sqrt{3}R_1^2}{4r} + \cdots\,,\\
b(r) &= \frac{r}{2\sqrt{3}} + q_1 R_1 + \frac{3\sqrt{3}R_1^2}{4r} + \cdots\,,\\
c(r) &= R_1 - \frac{9R_1^3}{r^2} + \frac{(27+36\sqrt{3}q_1)R_1^4}{r^3} + \cdots\,,\\
d(r) &= \frac{r}{3} + \frac{R_1}{2\sqrt{3}}(4q_1+\sqrt{3}) + \frac{3R_1^2}{r} + \cdots\,,
\end{aligned}
\tag{4.6}
$$

where q_1 and R_1 are constants that will be functions of of q_0 and R_0. Note that $c(r)$ is stabilised whilst the others diverge linearly. The expressions are needed to second order because we will be interested in the subleading terms of various integrals.

4.2.2 Commuting U(1) isometries and membrane configurations

The metrics (4.2) have three commuting U(1) isometries. Using the Euler coordinates (4.3), these can canonically be taken to be generated by $\partial_{\phi_1} \subset \mathrm{SU}(2)_L^\sigma$, $\partial_{\phi_2} \subset \mathrm{SU}(2)_L^\Sigma$ and $\partial_{\psi_1} + \partial_{\psi_2} \subset \mathrm{SU}(2)_R^D$.

The existence of three commuting U(1) isometries is very useful for considering rotating membranes. By placing the directions of rotation and wrapping along these U(1)s, most of the equations of motion are trivially satisfied as a statement of conserved charges. The remaining equation of motion for the radial direction then follows from a first order gauge fixing constraint, as discussed above.

However, the canonical U(1)s are not the most sensible for our purposes. Consider the redefinitions

$$
\begin{aligned}
\psi_3 &= \psi_1 + \psi_2\,, \qquad & \psi_4 &= \psi_1 - \psi_2\,,\\
\phi_3 &= \phi_1 + \phi_2\,, \qquad & \phi_4 &= \phi_1 - \phi_2\,.
\end{aligned}
\tag{4.7}
$$

Note that ψ_3, ψ_4 now have a range of 8π whilst ϕ_3, ϕ_4 have a range of 4π. Three commuting isometries now are $\partial_{\phi_3}, \partial_{\phi_4}, \partial_{\psi_3}$. As we shall see shortly, the first two U(1)s will now be contained in S^3s that do collapse and do not collapse, respectively, at the origin. In the IIA brane picture, the S^3 that does collapse is surrounding the brane whilst the S^3 that does

Target space coordinate	Configuration $\mathrm{I}_{\mathbb{B}}$	Configuration $\mathrm{II}_{\mathbb{B}}$	Configuration $\mathrm{III}_{\mathbb{B}}$
$\phi_3 =$	$\omega\tau$	$\lambda\delta$	$\omega\tau$
$\phi_4 =$	$\nu_2\tau$	$\nu_2\tau$	$\lambda\delta$
$\psi_3 =$	$\lambda\delta$	$\nu_3\tau$	$\nu_3\tau$
$t =$	$\kappa\tau$		
$r =$	$r(\sigma)$		

Table 1: Three rotating membrane configurations on the $\mathbb{B}_7$ metrics.

not collapse is inside the brane. In order for the dual field theory to be four dimensional, one must consider energies such that the finite S^3 is not probed. The charge generated by rotations along ∂_{ϕ_3}, inside the brane, will be denoted K_1, whilst the charge generated by rotations along ∂_{ϕ_4}, outside the brane, will be denoted by K_2. In the $\mathbb{B}_7$ family, the U(1) generated by ∂_{ψ_3}, the circle that is stabilised at infinity, is contained within the collapsing S^3 at the origin. Call this charge K_3. Note that the isometries transverse to the membrane do not have the interpretation of R-charge because the theory is $\mathcal{N} = 1$. We cannot have all three charges at once, as we need to use one of the isometries to wrap the membrane. This last point is necessary for the wrapping direction to drop out of the action integral.

The configuration of the membrane is taken to be trivial in the remaining directions

$$x = y = z = 0\,, \qquad \theta_1 = \theta_2 = \frac{\pi}{2}\,, \qquad \psi_4 = 0\,. \tag{4.8}$$

One can also extend the string in the xyz plane and indeed such configuration will be considered in a later section. There are then three possible configurations for the nontrivial directions, shown in table 1.

If the stabilised circle generated by ∂_{ψ_3} is considered as the M-theory circle, then configurations $\mathrm{II}_{\mathbb{B}}$ and $\mathrm{III}_{\mathbb{B}}$ reduce to rotating D2-branes or a D0-D2 state, depending on whether there is a rotation along the M-theory circle or not, whilst the $\mathrm{I}_{\mathbb{B}}$ configuration reduces to a rotating fundamental string.

At this point, we need to take into account the $\mathbb{Z}_N$ quotient of the G_2 manifold that was mentioned above. The effect of this quotient is to send

$$\psi_3 \to \frac{\psi_3}{N}\,. \tag{4.9}$$

The target space metric that is seen by the membrane is thus

$$\frac{1}{l_{11}^2} ds_{M2}^2 = -dt^2 + dr^2 + \frac{c(r)^2}{N^2} d\psi_3^2 + b(r)^2 d\phi_3^2 + a(r)^2 d\phi_4^2\,. \tag{4.10}$$

It is easily checked that the $\gamma_{0\alpha}$ constraints in (2.3) are satisfied. The remaining constraint, choosing the free constant $L = 1/\lambda$, then implies that

$$\left(\frac{dr}{d\sigma}\right)^2 = \begin{cases} \frac{\kappa^2 - b(r)^2\omega^2 - a(r)^2\nu_2^2}{c(r)^2 l_{11}^2/N^2} & (\text{Case } \mathrm{I}_{\mathbb{B}}) \\ \frac{\kappa^2 - a(r)^2\nu_2^2 - \nu_3^2 c(r)^2/N^2}{b(r)^2 l_{11}^2} & (\text{Case } \mathrm{II}_{\mathbb{B}}) \\ \frac{\kappa^2 - b(r)^2\omega^2 - c(r)^2\nu_2^2/N^2}{a(r)^2 l_{11}^2} & (\text{Case } \mathrm{III}_{\mathbb{B}}) \end{cases} \tag{4.11}$$

A further constraint must be imposed, this is the condition that the membrane doubles back on itself at some radius

$$\left.\frac{dr}{d\sigma}\right|_{r_0} = 0\,. \tag{4.12}$$

This condition gives a relationship between $(r_0, \kappa, \omega, \nu_2, \nu_3)$. We will use this relationship to eliminate κ below.

Also, one needs to impose a normalisation condition

$$2\pi = \int d\sigma = \int_0^{r_0} dr \frac{d\sigma}{dr}\,. \tag{4.13}$$

This gives an integral constraint between $(r_0, \kappa, \omega, \nu_2, \nu_3)$.

Other configurations are possible, in which the rotating directions or the wrapped direction is some linear combination of the U(1)s. However, these configurations will not generically satisfy the constraints, because the induced metric will have cross terms. Another possibility is to take different U(1) subgroups of the original SU(2)s. The present choices would appear to be the most natural and we will not consider other subgroups here.

Before moving on, one must check the consistency of the ansätze described here. Checking the equations of motion, we find that indeed all are consistent.

4.2.3 Energy and other conserved charges

The following conserved charges are naturally associated with the configuration

$$\begin{aligned} E &= -\frac{\delta I}{\delta \kappa}\,, & K_1 &= \frac{\delta I}{\delta \omega}\,, \\ K_2 &= \frac{\delta I}{\delta \nu_2}\,, & K_3 &= \frac{\delta I}{\delta \nu_3}\,, \end{aligned} \tag{4.14}$$

where I is the action (2.1). The κ derivative is taken at fixed $(r_0, \omega, \nu_2, \nu_3)$, and similarly for the other derivatives.

Let us do this in the three cases. Note that in passing from an integral over σ to an integral over r we multiply by four because of the periodicity of the integrand and the fact that the membrane doubles back on itself. We use the constraint (4.12) to eliminate κ. The different numerical factor in the different cases is due to the different ranges of the angle about which the membrane is wrapped.

- Case $\mathrm{I}_{\mathbb{B}}$

$$I = \frac{-32\pi}{N(2\pi)^2} \int d\tau \int_0^{r_0} dr |c(r)| \sqrt{\kappa^2 - b(r)^2\omega^2 - a(r)^2\nu_2^2}\,, \tag{4.15}$$

$$E = \frac{32\pi}{N(2\pi)^2} \int_0^{r_0} dr \frac{|c(r)|\sqrt{\omega^2 b(r_0)^2 + \nu_2^2 a(r_0)^2}}{\sqrt{\omega^2\left[b(r_0)^2 - b(r)^2\right] + \nu_2^2\left[a(r_0)^2 - a(r)^2\right]}}\,, \tag{4.16}$$

$$K_1 = \frac{32\pi}{N(2\pi)^2} \int_0^{r_0} dr \frac{\omega|c(r)|b(r)^2}{\sqrt{\omega^2\left[b(r_0)^2 - b(r)^2\right] + \nu_2^2\left[a(r_0)^2 - a(r)^2\right]}}\,, \tag{4.17}$$

$$K_2 = \frac{32\pi}{N(2\pi)^2} \int_0^{r_0} dr \frac{\nu_2|c(r)|a(r)^2}{\sqrt{\omega^2\left[b(r_0)^2 - b(r)^2\right] + \nu_2^2\left[a(r_0)^2 - a(r)^2\right]}}\,, \tag{4.18}$$

$$K_3 = 0\,. \tag{4.19}$$

- Case $\mathrm{II}_{\mathbb{B}}$

$$I = \frac{-16\pi}{(2\pi)^2}\int d\tau \int_0^{r_0} dr|b(r)|\sqrt{\kappa^2 - a(r)^2\nu_2^2 - \frac{\nu_3^2 c(r)^2}{N^2}}\,, \tag{4.20}$$

$$E = \frac{16\pi}{(2\pi)^2}\int_0^{r_0} dr \frac{|b(r)|\sqrt{\nu_2^2 a(r_0)^2 + \nu_3^2 c(r_0)^2/N^2}}{\sqrt{\nu_2^2\left[a(r_0)^2 - a(r)^2\right] + \nu_3^2\left[c(r_0)^2 - c(r)^2\right]/N^2}}\,, \tag{4.21}$$

$$K_1 = 0, \tag{4.22}$$

$$K_2 = \frac{16\pi}{(2\pi)^2}\int_0^{r_0} dr \frac{\nu_2|b(r)|a(r)^2}{\sqrt{\nu_2^2\left[a(r_0)^2 - a(r)^2\right] + \nu_3^2\left[c(r_0)^2 - c(r)^2\right]/N^2}}\,, \tag{4.23}$$

$$K_3 = \frac{16\pi}{N^2(2\pi)^2}\int_0^{r_0} dr \frac{\nu_3|b(r)|c(r)^2}{\sqrt{\nu_2^2\left[a(r_0)^2 - a(r)^2\right] + \nu_3^2\left[c(r_0)^2 - c(r)^2\right]/N^2}}\,. \tag{4.24}$$

- Case $\mathrm{III}_{\mathbb{B}}$

$$I = \frac{-16\pi}{(2\pi)^2}\int d\tau \int_0^{r_0} dr|a(r)|\sqrt{\kappa^2 - b(r)^2\omega^2 - \frac{\nu_3^2 c(r)^2}{N^2}}\,, \tag{4.25}$$

$$E = \frac{16\pi}{(2\pi)^2}\int_0^{r_0} dr \frac{|a(r)|\sqrt{\omega^2 b(r_0)^2 + \nu_3^2 c(r_0)^2/N^2}}{\sqrt{\omega^2\left[b(r_0)^2 - b(r)^2\right] + \nu_3^2\left[c(r_0)^2 - c(r)^2\right]/N^2}}\,, \tag{4.26}$$

$$K_1 = \frac{16\pi}{(2\pi)^2}\int_0^{r_0} dr \frac{\omega|a(r)|b(r)^2}{\sqrt{\omega^2\left[b(r_0)^2 - b(r)^2\right] + \nu_3^2\left[c(r_0)^2 - c(r)^2\right]/N^2}}\,, \tag{4.27}$$

$$K_2 = 0\,, \tag{4.28}$$

$$K_3 = \frac{16\pi}{N^2(2\pi)^2}\int_0^{r_0} dr \frac{\nu_3|a(r)|c(r)^2}{\sqrt{\omega^2\left[b(r_0)^2 - b(r)^2\right] + \nu_3^2\left[c(r_0)^2 - c(r)^2\right]/N^2}}\,. \tag{4.29}$$

These integrals may be expanded for large and small r_0 using the expansions (4.5) and (4.6). In the various integrals there are usually two constants, such as ω and ν_2 in the $\mathrm{I}_{\mathbb{B}}$ case. These are nontrivially related through the normalisation constraint (4.13). Here we will only consider the cases where one of the constants is zero, corresponding to a rotation in only one direction. In these cases we see that the remaining constant drops out of the integral and the normalisation constraint does not need to be evaluated.

For short membranes, small r_0, one may use the Taylor expansions about the origin to evaluate the integrals. For long membranes, large r_0, one may only use the expansions about infinity to evaluate the integral if the integral is diverging with r_0 because in this case the integral is dominated by the contributions at infinity. One then needs to check that there is not an r_0 contribution from the interior of the integrand. Naively, the integrals for large r_0 are done as follows

$$\begin{aligned}\int_0^{r_0} f(r,r_0)dr &\approx \int_\Lambda^{r_0} f(r,r_0)dr = r_0\int_\Lambda^1 f(ur_0,r_0)du \\ &\approx r_0\int_\Lambda^1\left[F_m(u)r_0^m + F_{m-1}(u)r_0^{m-1} + \cdots\right]du\,,\end{aligned} \tag{4.30}$$

where Λ is some cutoff and we ignore contributions from this end of the integral. The final expression represents an expansion of $f(ur_0,r_0)$ about $r_0=\infty$. In the final result of

Configuration	$r_0 \to 0$ (short membranes)	$r_0 \to \infty$ (long membranes)
$\mathrm{I}_{\mathbb{B}}$, $\nu_2 = 0$	$E = kN^{-1/3}K_1^{2/3} + \cdots$	$E = kR_1^{1/2}N^{-1/2}K_1^{1/2} + \cdots$
$\mathrm{I}_{\mathbb{B}}$, $\omega = 0$	$E - \dfrac{K_2}{R_0} = -k\dfrac{N^2K_2^3}{R_0^7} + \cdots$	$E = kR_1^{1/2}N^{-1/2}K_2^{1/2} + \cdots$
$\mathrm{II}_{\mathbb{B}}$, $\nu_2 = 0$	$E = kN^{2/3}K_3^{2/3} + \cdots$	$E - \dfrac{NK_3}{R_1} = kR_1N^{1/3}K_3^{1/3} + \cdots$
$\mathrm{II}_{\mathbb{B}}$, $\nu_3 = 0$	$E - \dfrac{K_2}{R_0} = -k\dfrac{K_2^3}{R_0^7} + \cdots$	$E = kK_2^{2/3} + \cdots$
$\mathrm{III}_{\mathbb{B}}$, $\nu_3 = 0$	$E = kR_0^{1/2}K_1^{1/2} + \cdots$	$E = kK_1^{2/3} + \cdots$
$\mathrm{III}_{\mathbb{B}}$, $\omega = 0$	$E = kR_0^{1/2}N^{1/2}K_3^{1/2} + \cdots$	$E - \dfrac{NK_3}{R_1} = kR_1N^{1/3}K_3^{1/3} + \cdots$

Table 2: Energy - Charge relations for membranes on $\mathbb{B}_7$ metrics.

this calculation, we may trust any terms that diverge as $r_0 \to \infty$. One thing that may go wrong is that the leading order coefficient, $F_m(u)$, in the final equation of (4.30) integrates to zero, meaning that there is no r_0^m power term. In this case one should do the full integral numerically to check whether the vanishing is a result of power expanding inside the integral, and see what the leading order coefficient is. Alternatively one can try to do the integral exactly without expanding the integrand fully. Doing this is crucial to obtain the logarithmic term in the next subsection.

Given the resulting expressions for E and the Ks, one then eliminates r_0 to obtain the results of table 2. In this table k is used to denote positive numerical factors. Dependence on R_0, R_1, N is kept explicit. It turns out there is no dependence on q_0, q_1 to the order considered in the table.

The results in table 2 have a physical interpretation. Note that there are four types of leading order behaviour. Use K to denote a generic charge and R to denote either R_1 or R_2.

- $E = kR^{1/2}K^{1/2}$: this is the well known Regge relation for strings in flat space. It arises when the δ-direction of the membrane is wrapped on a stabilised U(1) and when the direction of rotation is a U(1) that is not stabilised (i.e. collapsing if we are at the origin or expanding if we are going to infinity).

- $E = kK^{2/3}$: this is the result for membranes rotating in flat space. It arises when neither the δ-direction nor the direction of rotation is stabilised.

- $E - \frac{K}{R} = kRK^{1/3}$: this result arises for long strings when the δ-direction is not stabilised, but the direction of rotation is stabilised. Interestingly, this relation was also observed in a different configuration [13] in $AdS_7 \times S^4$, suggesting perhaps that it is quite generic.

- $E - \frac{K}{R} = -k\frac{K^3}{R^7}$: this result arises for short strings when the δ-direction collapses, but the direction of rotation does not collapse.

The behaviour of the energy-charge relationship would thus seem to depend on whether the wrapped circle and the circle of rotation are stabilised. In the above configurations one case is missing, there is no case in which both the δ-circle and the circle of rotation do not collapse. For short strings, we will find such a configuration in the $\mathbb{D}_7$ metrics below, because more circles are non-shrinking at the origin. However, within the set of configurations we have considered thus far, we cannot find a configuration in which two circles are stabilised at infinity, because the G_2 metrics only have one stabilised circle. We might expect such a configuration to give logarithms by analogy with the previous section when we considered membranes rotating on $AdS_4 \times S^7$: to move from a relationship of the form $E - K = K^{1/3}$ to a relationship $E - K = \ln K$, we changed the wrapped circle to make it stabilised. To achieve this in the present case, we need to use the non-compact directions. However, due to the tension of the membrane, one cannot simply have a closed membrane in flat space. The resolution is to consider an infinite membrane in the non-compact directions and study the energy density of the configuration. Insofar as the equations are concerned, this is effectively the same as wrapping the membane.

4.2.4 Using the non-compact directions: logarithms

Writing the eleven dimensional metric as

$$\frac{1}{l_{11}^2} ds_{11}^2 = -dt^2 + dx^2 + dy^2 + dz^2 + ds_7^2 , \tag{4.31}$$

the following configuration, which we denote $\mathrm{IV}_{\mathbb{B}}$, has the desired feature of having both a wrapped and a rotating direction asymptotically stabilised. The nontrivial coordiantes are

$$t = \kappa\tau , \quad \psi_3 = \nu_3 \tau , \quad r = r(\sigma) , \quad x = \lambda\delta . \tag{4.32}$$

The remaining coordinates are trivial

$$\phi_3 = \phi_4 = 0 , \quad y = z = 0 , \quad \theta_1 = \theta_2 = \frac{\pi}{2} , \quad \psi_4 = 0 . \tag{4.33}$$

One might also want to consider having the rotation in the non-compact direction, but this seems to cause difficulties with the implementation of the endpoint constraint (4.12). The target space metric seen by the membrane now becomes

$$\frac{1}{l_{11}^2} ds_{M2}^2 = -dt^2 + dr^2 + \frac{c(r)^2}{N^2} d\psi_3^2 + dx^2 . \tag{4.34}$$

The action, energy and charge, *per unit length* along the non-compact x direction, are easily worked out to be

- Case $\mathrm{IV}_{\mathbb{B}}$

$$I = \frac{-8\pi}{(2\pi)^2} \int d\tau \int_0^{r_0} dr \sqrt{\kappa^2 - \frac{\nu_3^2 c(r)^2}{N^2}} , \tag{4.35}$$

$$E = \frac{8\pi}{(2\pi)^2}\int_0^{r_0} dr \frac{|c(r_0)|}{\sqrt{[c(r_0)^2 - c(r)^2]}}\,, \tag{4.36}$$

$$K_3 = \frac{8\pi}{N(2\pi)^2}\int_0^{r_0} dr \frac{c(r)^2}{\sqrt{[c(r_0)^2 - c(r)^2]}}. \tag{4.37}$$

In the large membrane limit, $r_0 \to \infty$, these integrals are dominated by their behaviour at r_0, thus we may expand the integrand and evaluate only at r_0. We substitute the expansion of $c(r)$ to second order into the integrand and evaluate the integral. Expanding the integrand fully before integrating will not give the correct answer, as no logs will appear.

The integrals give, to leading and subleading order

$$\begin{aligned} K_3 &= k\frac{-ir_0R_1}{2N}\left[3K\left(\sqrt{\frac{2r_0^2}{9R_1^2}-1}\right)+E\left(\sqrt{\frac{2r_0^2}{9R_1^2}-1}\right)\right], \\ E &= k\frac{ir_0}{2}\left[K\left(\sqrt{\frac{2r_0^2}{9R_1^2}-1}\right)-E\left(\sqrt{\frac{2r_0^2}{9R_1^2}-1}\right)\right], \end{aligned} \tag{4.38}$$

Where $K(x)$ and $E(x)$ are complete elliptic integrals of the first and second kind. The constants k in the above two lines are equal, but below we use k to denote any constant, with dependence on ρ_0 and R_1 kept explicit.

In order to evaluate these integrals we need the following expressions for the asymptotics as $x \to \infty$ of the elliptic integrals

$$\begin{aligned} K(\sqrt{x-1}) &\sim -\frac{i}{2}x^{-1/2}(\ln x + i\pi)\,, \\ E(\sqrt{x-1}) &\sim ix^{1/2}\,. \end{aligned} \tag{4.39}$$

These formulae are easily derived by expressing the complete elliptic integrals as hypergeometric functions and then using the Pfaff and Gauss theorems for hypergeometric functions [54].

Thus we have that whilst

$$K_3 = \frac{kr_0^2}{N} + \cdots, \tag{4.40}$$

the difference

$$E - \frac{NK_3}{R_1} = kR_1 \ln\frac{r_0}{R_1} + \cdots\,. \tag{4.41}$$

Combining these two expressions gives the new kind of behaviour for long membranes

$$E - \frac{NK_3}{R_1} = kR_1 \ln\frac{NK_3}{R_1^2} + \cdots\,. \tag{4.42}$$

This behaviour is different from the behaviours of the previous section because both the direction of wrapping and the direction of rotation are stabilised as we go to infinity.

For short membranes with this configuration, we get $E = kN^{1/2}K_3^{1/2} + \cdots$ as expected for a membrane where the δ-direction is stabilised but the rotation direction collapses at the origin. These solutions thus realise a transition from Regge behaviour for short membranes, to logarithmic behaviour for long membranes without finite size effects [8].

Another way of getting around the fact that a closed tubular membrane in flat space can't exist as a static solution due to the membrane tension is to consider a pulsating membrane solution, analogous to the well-known pulsating closed string solution. The solution will not be particularly straightforward to construct in the present context. Such a solution may have a tunable amplitude, in which case one could make the energy of the pulsations negligible compared with the energy of the rotation and therefore the calculations of this section will go through as the dominant effect.

4.3 Membranes on the $\mathbb{D}_7$ family

4.3.1 Metric formulae

The metrics can be written in the form

$$\begin{aligned} ds_7^2 = dr^2 + a(r)^2\left[(\Sigma_1+g(r)\sigma_1)^2+(\Sigma_2+g(r)\sigma_2)^2\right]+c(r)^2(\Sigma_3+g_3(r)\sigma_3)^2+ \\ +b(r)^2\left[\sigma_1^2+\sigma_2^2\right]+f(r)^2\sigma_3^2\,, \end{aligned} \tag{4.43}$$

where Σ_i, σ_i are left-invariant one-forms on the SU(2)s, as previously. The six functions are not all independent

$$g(r) = \frac{-a(r)f(r)}{2b(r)c(r)}\,, \qquad g_3(r) = -1+2g(r)^2\,. \tag{4.44}$$

None of the radial functions are known explicitly, although the asymptotics at the origin and at infinity are known. The asymptotics are found by finding Taylor series solutions to the first order equations for the radial functions. The equations are [50]

$$\begin{aligned} \dot{a} &= -\frac{c}{2a}+\frac{a^5f^2}{8b^4c^3}\,, & \dot{b} &= -\frac{c}{2b}-\frac{a^2(a^2-3c^2)f^2}{8b^3c^3}\,, \\ \dot{c} &= -1+\frac{c^2}{2a^2}+\frac{c^2}{2b^2}-\frac{3a^2f^2}{8b^4}\,, & \dot{f} &= -\frac{a^4f^3}{4b^4c^3}\,. \end{aligned} \tag{4.45}$$

As $r\to 0$ one has

$$\begin{aligned} a(r) &= \frac{r}{2}-\frac{(q_0^2+2)r^3}{288R_0^2}-\frac{(-74-29q_0^2+31q_0^4)r^5}{69120R_0^4}+\cdots\,, \\ b(r) &= R_0-\frac{(q_0^2-2)r^2}{16R_0}-\frac{(13-21q_0^2+11q_0^4)r^4}{1152R_0^3}+\cdots\,, \\ c(r) &= -\frac{r}{2}-\frac{(5q_0^2-8)r^3}{288R_0^2}-\frac{(232-353q_0^2+157q_0^4)r^5}{34560R_0^4}+\cdots\,, \\ f(r) &= q_0R_0+\frac{q_0^3r^2}{16R_0}+\frac{q_0^3(-14+11q_0^2)r^4}{1152R_0^3}+\cdots\,, \end{aligned} \tag{4.46}$$

where q_0 and R_0 are constants. Note that $a(r)$ and $c(r)$ collapse and the other two functions do not. As $r\to\infty$ we have

$$a(r) = \frac{r}{\sqrt{6}}-\frac{\sqrt{3}q_1R_1}{\sqrt{2}}+\frac{(27\sqrt{6}-96h_1)R_1^2}{96r}+\cdots\,,$$

$$b(r) = \frac{r}{\sqrt{6}} - \frac{\sqrt{3}q_1 R_1}{\sqrt{2}} + \frac{h_1 R_1^2}{r} + \cdots ,$$
$$c(r) = \frac{-r}{3} + q_1 R_1 - \frac{9R_1^2}{8r} + \cdots ,$$
$$f(r) = R_1 - \frac{27R_1^3}{8r^2} - \frac{81R_1^4 q_1}{4r^3} + \cdots . \tag{4.47}$$

With constants R_1, q_1, h_1. Note that $f(r)$ stabilises. Three constants appear to this order, whilst there were only two constants in the expansion about the origin. This just means that for some values of these constants, the corresponding solution will diverge before it reaches zero. In any case, we find no h_1 dependence in the results below.

4.3.2 Membrane configurations

The situation is essentially the same as for the $\mathbb{B}_7$ family of metrics. Again one has three commuting U(1) isometries, generated by $\partial_{\phi_1} \subset \mathrm{SU}(2)^{\sigma}_L$, $\partial_{\phi_2} \subset \mathrm{SU}(2)^{\Sigma}_L$ and $\partial_{\psi_1} + \partial_{\psi_2} \subset \mathrm{SU}(2)^D_R$. One difference, however, is that now ∂_{ϕ_1} generates a U(1) that does not collapse and ∂_{ϕ_2} generates a circle that does collapse, so there is no need to change variables to ϕ_3 and ϕ_4 as previously. In fact, such a change would not give a valid solution. We do however need to define

$$\psi_3 = \psi_1 + \psi_2 , \qquad \psi_4 = \psi_1 - \psi_2 . \tag{4.48}$$

Note that now ψ_3, ψ_4 have ranges of 8π whilst ϕ_1, ϕ_2 have ranges of 2π. Three commuting U(1) isometries are then ∂_{ϕ_1}, ∂_{ϕ_2} and ∂_{ψ_3}. There are no branes in reduction of these configurations to well-defined IIA solutions. The circle generated by ∂_{ϕ_1} and ∂_{ψ_3} do not collapse in the interior and thus rotation in these directions corresponds to charges, K_1 and K_2 respectively. The ∂_{ϕ_2} circle does collapse and rotation about this circle will give a charge denoted by K_3.

Similar to before, we take

$$x = y = z = 0 , \qquad \theta_1 = \theta_2 = \frac{\pi}{2} , \qquad \psi_4 = \frac{\pi}{2} . \tag{4.49}$$

Note that the value of ψ_4 is different. This value is needed to diagonalise the metric seen by the membrane and hence satisfy the constraints. As in the previous subsection, there are three possible configurations for the nontrivial directions, shown in table 3.

Target space coordinate	Configuration $\mathrm{I}_{\mathbb{D}}$	Configuration $\mathrm{II}_{\mathbb{D}}$	Configuration $\mathrm{III}_{\mathbb{D}}$
$\phi_1 =$	$\omega_1 \tau$	$\lambda\delta$	$\omega_1 \tau$
$\phi_2 =$	$\nu\tau$	$\nu\tau$	$\lambda\delta$
$\psi_3 =$	$\lambda\delta$	$\omega_2\tau$	$\omega_2\tau$
$t =$	$\kappa\tau$		
$r =$	$r(\sigma)$		

Table 3: Three rotating membrane configurations on the $\mathbb{D}_7$ metrics.

The target space metric that is seen by the membrane, after doing the $\mathbb{Z}_N$ quotient on ψ_3, is

$$
\begin{aligned}
\frac{1}{l_{11}^2} ds_{M2}^2 &= -dt^2 + dr^2 + \left[\frac{1}{4} f(r)^2 + c(r)^2 g(r)^4\right] \frac{d\psi_3^2}{N^2} + \left[a(r)^2 g(r)^2 + b(r)^2\right] d\phi_1^2 + a(r)^2 d\phi_2^2 \\
&\equiv -dt^2 + dr^2 + \frac{C(r)^2}{N^2} d\psi_3^2 + B(r)^2 d\phi_1^2 + A(r)^2 d\phi_2^2\,, \qquad (4.50)
\end{aligned}
$$

where we have introduced functions $A(r), B(r), C(r)$ in order to make the following formulae less ugly. The asymptotics for these functions follow from the limits (4.46) and (4.47) and the algebraic equations in (4.45). As $r \to 0$ we have

$$
\begin{aligned}
A(r) &= \frac{r}{2} - \frac{(q_0^2+2)r^3}{288 R_0^2} - \frac{(-74 - 29q^2 + 31q^4)r^5}{69120 R^4} + \cdots\,, \\
B(r) &= R_0 + \frac{(4-q_0^2)r^2}{32 R_0} - \frac{(61q^4 - 152q^2 + 208)r^4}{18432 R^3} + \cdots\,, \\
C(r) &= \frac{q_0 R_0}{2} + \frac{3q_0^3 r^2}{64 R_0} - \frac{(-160 + 121q^2)q^3 r^4}{12288 R^3} + \cdots\,, \qquad (4.51)
\end{aligned}
$$

and as $r \to \infty$ we have

$$
\begin{aligned}
A(r) &= \frac{r}{\sqrt{6}} - \frac{\sqrt{3} q_1 R_1}{\sqrt{2}} + \frac{(27\sqrt{6} - 96h_1)R_1^2}{96r} + \cdots\,, \\
B(r) &= \frac{r}{\sqrt{6}} - \frac{\sqrt{3} q_1 R_1}{\sqrt{2}} + \frac{(18\sqrt{6} + 96h_1)R_1^2}{96r} + \cdots\,, \\
C(r) &= \frac{R_1}{2} - \frac{9R_1^3}{8r^2} - \frac{27 q_1 R_1^4}{4r^3} + \cdots\,. \qquad (4.52)
\end{aligned}
$$

The only nontrivial constraint from (2.3) now implies that

$$
\left(\frac{dr}{d\sigma}\right)^2 = \begin{cases} \frac{\kappa^2 - B(r)^2\omega_1^2 - A(r)^2\nu^2}{C(r)^2 l_{11}^2/N^2} & (\text{Case I}_{\mathbb{D}}) \\ \frac{\kappa^2 - A(r)^2\nu^2 - C(r)^2\omega_2^2/N^2}{B(r)^2 l_{11}^2} & (\text{Case II}_{\mathbb{D}}) \\ \frac{\kappa^2 - B(r)^2\omega_1^2 - C(r)^2\omega_2^2/N^2}{A(r)^2 l_{11}^2} & (\text{Case III}_{\mathbb{D}}) \end{cases} \qquad (4.53)
$$

The issue of consistency is more subtle in these cases than in the $\mathbb{B}_7$ cases. This is because there are more cross terms in the metric. It turns out that in order for the ansätze to be consistent, one can only have one of the angular momenta to be nonzero in any given solution. Thus, for example, in the type-$\text{I}_{\mathbb{D}}$ solution one must have either $\omega_1 = 0$ or $\nu = 0$ in order for the configuration to solve the equations of motion. These are the configurations we shall consider below.

4.3.3 Energy and other conserved charges

Again, the differing numerical factors in the expressions below are due to differences in the ranges of angles.

- Case $\mathrm{I}_{\mathbb{D}}$

$$I = \frac{-32\pi}{N(2\pi)^2}\int d\tau \int_0^{r_0} dr|C(r)|\sqrt{\kappa^2 - B(r)^2\omega_1^2 - A(r)^2\nu^2}\,, \tag{4.54}$$

$$E = \frac{32\pi}{N(2\pi)^2}\int_0^{r_0} dr \frac{|C(r)|\sqrt{\omega_1^2 B(r_0)^2 + \nu^2 A(r_0)^2}}{\sqrt{\omega_1^2\left[B(r_0)^2 - B(r)^2\right] + \nu^2\left[A(r_0)^2 - A(r)^2\right]}}\,, \tag{4.55}$$

$$K_1 = \frac{32\pi}{N(2\pi)^2}\int_0^{r_0} dr \frac{\omega_1|C(r)|B(r)^2}{\sqrt{\omega_1^2\left[B(r_0)^2 - B(r)^2\right] + \nu^2\left[A(r_0)^2 - A(r)^2\right]}}\,, \tag{4.56}$$

$$K_2 = 0\,, \tag{4.57}$$

$$K_3 = \frac{32\pi}{N(2\pi)^2}\int_0^{r_0} dr \frac{\nu|C(r)|A(r)^2}{\sqrt{\omega_1^2\left[B(r_0)^2 - B(r)^2\right] + \nu^2\left[A(r_0)^2 - A(r)^2\right]}}\,. \tag{4.58}$$

- Case $\mathrm{II}_{\mathbb{D}}$

$$I = \frac{-8\pi}{(2\pi)^2}\int d\tau \int_0^{r_0} dr|B(r)|\sqrt{\kappa^2 - A(r)^2\nu^2 - \frac{\omega_2^2 C(r)^2}{N^2}}\,, \tag{4.59}$$

$$E = \frac{8\pi}{(2\pi)^2}\int_0^{r_0} dr \frac{|B(r)|\sqrt{\nu^2 A(r_0)^2 + \omega_2^2 C(r_0)^2/N^2}}{\sqrt{\nu^2\left[A(r_0)^2 - A(r)^2\right] + \omega_2^2\left[C(r_0)^2 - C(r)^2\right]/N^2}}\,, \tag{4.60}$$

$$K_1 = 0\,, \tag{4.61}$$

$$K_2 = \frac{8\pi}{N^2(2\pi)^2}\int_0^{r_0} dr \frac{\omega_2|B(r)|C(r)^2}{\sqrt{\nu^2\left[A(r_0)^2 - A(r)^2\right] + \omega_2^2\left[C(r_0)^2 - C(r)^2\right]/N^2}}\,, \tag{4.62}$$

$$K_3 = \frac{8\pi}{(2\pi)^2}\int_0^{r_0} dr \frac{\nu|B(r)|A(r)^2}{\sqrt{\nu^2\left[A(r_0)^2 - A(r)^2\right] + \omega_2^2\left[C(r_0)^2 - C(r)^2\right]/N^2}}\,. \tag{4.63}$$

- Case $\mathrm{III}_{\mathbb{D}}$

$$I = \frac{-8\pi}{(2\pi)^2}\int d\tau \int_0^{r_0} dr|A(r)|\sqrt{\kappa^2 - B(r)^2\omega_1^2 - \frac{\omega_2^2 C(r)^2}{N^2}}\,, \tag{4.64}$$

$$E = \frac{8\pi}{(2\pi)^2}\int_0^{r_0} dr \frac{|A(r)|\sqrt{\omega_1^2 B(r_0)^2 + \omega_2^2 C(r_0)^2/N^2}}{\sqrt{\omega_1^2\left[B(r_0)^2 - B(r)^2\right] + \omega_2^2\left[C(r_0)^2 - C(r)^2\right]/N^2}}, \tag{4.65}$$

$$K_1 = \frac{8\pi}{(2\pi)^2}\int_0^{r_0} dr \frac{\omega_1|A(r)|B(r)^2}{\sqrt{\omega_1^2\left[B(r_0)^2 - B(r)^2\right] + \omega_2^2\left[C(r_0)^2 - C(r)^2\right]/N^2}}, \tag{4.66}$$

$$K_2 = \frac{8\pi}{N^2(2\pi)^2}\int_0^{r_0} dr \frac{\omega_2|A(r)|C(r)^2}{\sqrt{\omega_1^2\left[B(r_0)^2 - B(r)^2\right] + \omega_2^2\left[C(r_0)^2 - C(r)^2\right]/N^2}}, \tag{4.67}$$

$$K_3 = 0\,. \tag{4.68}$$

These integrals are now performed in the small and large membrane limits in the same way as for the $\mathbb{B}_7$ metrics. The results are presented in table 4. Again, k denotes positive numerical constants, with dependence on R_0, R_1, q_0, q_1, N kept explicit. It is not surprising that a dependence on q_0 now emerges because, unlike the $\mathbb{B}_7$ cases, the principle interpretation of this parameter is as measuring the squashing of the bolt at the origin [50].

The behaviours observed are the same as for the $\mathbb{B}_7$ metrics, except that there is one new possibility for short strings

Configuration	$r_0 \to 0$	$r_0 \to \infty$
$\mathrm{I}_{\mathbb{D}}$, $\omega_1 = 0$	$E = kR_0^{1/2} q_0^{1/2} N^{-1/2} K_3^{1/2} + \cdots$	$E = kR_1^{1/2} N^{-1/2} K_3^{1/2} + \cdots$
$\mathrm{I}_{\mathbb{D}}$, $\nu = 0$	$E = \frac{K_1}{R_0} = kR_0^2 \frac{q_0}{N\sqrt{4-q_0^2}} + \cdots$	$E = kR_1^{1/2} N^{-1/2} K_1^{1/2} + \cdots$
$\mathrm{II}_{\mathbb{D}}$, $\omega_2 = 0$	$E = kR_0^{1/2} K_3^{1/2} + \cdots$	$E = kK_3^{2/3} + \cdots$
$\mathrm{II}_{\mathbb{D}}$, $\nu = 0$	$E = \frac{2NK_2}{R_0 q_0} = k\frac{R_0^2}{q_0} + \cdots$	$E - \frac{2NK_2}{R_1} = kR_1 N^{1/3} K_2^{1/3} + \cdots$
$\mathrm{III}_{\mathbb{D}}$, $\omega_2 = 0$	$E - \frac{K_1}{R_0} = -k\frac{(4-q_0^2)^2 K_1^3}{R_0^7} + \cdots$	$E = kK_1^{2/3} + \cdots$
$\mathrm{III}_{\mathbb{D}}$, $\omega_1 = 0$	$E - \frac{2NK_2}{q_0 R_0} = -k\frac{q_0 N^3 K_2^3}{R_0^7} + \cdots$	$E - \frac{2NK_2}{R_1} = kR_1 N^{1/3} K_2^{1/3} + \cdots$

Table 4: Energy - Charge relations for membranes on $\mathbb{D}_7$ metrics.

- $E = \frac{K}{R} = kR^2$: This arises when the δ-direction does not collapse at the origin, so the rotation is string-like, and the direction of rotation also does not collapse. There will be a dependence on q.

4.3.4 Using the non-compact directions again: logarithms

Writing the eleven dimensional metric as

$$\frac{1}{l_{11}^2} ds_{11}^2 = -dt^2 + dx^2 + dy^2 + dz^2 + ds_7^2 \,, \tag{4.69}$$

we may do exactly the same calcutions as we did before for the $\mathbb{B}_7$ metrics. The configuration that follows will be denoted $\mathrm{IV}_{\mathbb{D}}$,

$$\begin{aligned} &t = \kappa\tau\,, \qquad \psi_3 = \omega_2\tau\,, \quad \phi_1 = \phi_2 = 0\,, \qquad r = r(\sigma)\,, \\ &y = z = 0\,, \qquad x = 0\,, \qquad \phi = \lambda\delta\,, \qquad \theta_1 = \theta_2 = \frac{\pi}{2}\,, \qquad \psi_4 = \frac{\pi}{2}\,. \end{aligned} \tag{4.70}$$

The target space metric seen by the membrane now becomes

$$\frac{1}{l_{11}^2} ds_{M2}^2 = -dt^2 + dr^2 + \frac{C(r)^2}{N^2} d\psi_3^2 + dx^2 \,. \tag{4.71}$$

The action, energy and charge, per unit length along the x direction, are easily worked out to be

- Case $\mathrm{IV}_{\mathbb{D}}$

$$I = \frac{-8\pi}{(2\pi)^2} \int d\tau \int_0^{r_0} dr \sqrt{\kappa^2 - \frac{\omega_2^2 C(r)^2}{N^2}}\,, \tag{4.72}$$

$$E = \frac{8\pi}{(2\pi)^2} \int_0^{r_0} dr \frac{|C(r_0)|}{\sqrt{[C(r_0)^2 - C(r)^2]}}\,, \tag{4.73}$$

$$K_3 = \frac{8\pi}{N(2\pi)^2} \int_0^{r_0} dr \frac{C(r)^2}{\sqrt{[C(r_0)^2 - C(r)^2]}}\,. \tag{4.74}$$

Obviously, this is exactly the same as the $\mathbb{B}_7$ case but with $c(r) \to C(r)$. The asymptotic expansions to second order are the same for $c(r)$ and $C(r)$ if we let $R_1 \to R_1/2$. Thus we get the same result

$$E - \frac{2NK_3}{R_1} = kR_1 \ln \frac{NK_3}{R_1^2} + \cdots . \tag{4.75}$$

For short membranes with this configuration, we get $E = 2NK_3/(R_1 q_1) = kR_1/q + \cdots$. As commented before, we might expect to also find a pulsating membrane solution with these energy - charge relations.

5. Discussion, comments regarding dual operators and open issues

Let us first review what we have done and found in this work. Motivated by the recent developments mentioned in the first section, we studied membranes rotating in different geometries that are of interest as duals to gauge theories.

To start with, we considered an $AdS_4 \times M_7$ spacetime. Depending on the holonomy of M_7, these are dual to $2+1$ dimensional conformal field theories with a varying number of preserved supersymmetries. In all of these manifolds, we found that rotating membrane configurations may develop relations for the energy E, spin S, and R-symmetry angular momentum J, of the form $E - S \sim \ln S$ and $E - J - S \sim 1/J \ln^2(S/J)$, as had previously been found for strings on various backgrounds. The same logarithmic results were found for membranes moving in a warped $AdS_5 \times M_6$ geometry that is dual to a four dimensional $N = 2$ conformal field theory. We also recovered previous non-logarithmic results for membranes and explained the difference between the logarithmic and non-logarithmic cases in terms of whether the direction wrapped by the membrane was stabilised at infinity or not.

According to the correspondence between high angular momentum strings/membranes and 'long' operators [1], these rotating membranes should be dual to certain twist two operators in the corresponding conformal field theory that have anomalous dimensions given by the relation between energy (or conformal dimension), spin and J-charge calculated on the gravity side of the duality. These results point to the fact that for geometries of the form $AdS_p \times M_q$, it will be possible to find membrane/string configurations dual to 'long' twist two operators.

Given these results, we were lead to a very natural extension; geometries that are not of the form $AdS_p \times M_q$. Section 4 of this work presents a very detailed study of membranes rotating in M-theory backgrounds of the form $\mathbb{R}^{1,3} \times M_7$, where M_7 is now a non-compact G_2 holonomy manifold. These backgrounds are thought to be dual to $\mathcal{N} = 1$ SYM, which is a confining 'QCD-like' theory.

The results of section four could be summarised as follows. We have found rotating membrane configurations that should be dual to operators with energy-angular momentum relations, using K to denote the angular momentum/dual charge, of the following form for small quantum numbers $E \sim K^{2/3}$, $E - K \sim K^3$, $E \sim K^{1/2}$, $E \sim K =$ constant. When continued to the large quantum number regime these may become $E \sim K^{1/2}$, $E \sim K^{2/3}$, $E - K \sim K^{1/3}$, $E - K \sim \ln K$. For the logarithmic cases, we considered energy and charge densities of a noncompact membrane. Some of these configurations seem to realise the proposal of [8] for rotating solutions in a confining geometry to exhibit a transition for Regge-like to D.I.S.-like behaviour without finite size effects.

Several comments are in order. First of all, we consider these results to be interesting. Not many dynamical or quantitative tests of the duality between M theory on G_2 manifolds and $\mathcal{N} = 1$ SYM theory seem to exist. We hope that our results are a step towards an understanding of the duality that involves both a dynamical and a quantitative statement. Indeed, the fact that we obtained results that look very much like they should correspond to anomalous dimensions of operators, suggests that the energy of gravity states corresponds to the dimension of gauge theory operators. This is not at all otherwise obvious, given the lack of conformal symmetry and the lack of a holographic formulation of the duality that explicitly links bulk states with boundary operators.

We should point out that our results leave many issues open. These issues seem to be inextricably tied up with limitations of current understanding of the duality. To begin with, due to the running of the gauge theory dual coupling, it is not evident how to read off from our solutions the 't Hooft coupling dependence of the relation between energy/dimension and spin/charge in the field theory. Then, the interpretation of the field theory dual charge to the angular momenta of the membrane is not totally clear. Given that the rotations are not in the four flat non-compact directions of the spacetime, it is not obvious why the charge should be four dimensional lorentzian spin and if it is not spin, then it is not clear what else it could be. As well as the known behaviours, like $E \sim K^{1/2}$, $E \sim K^{2/3}$, which are Regge-like behaviours for short membranes, and $E - K \sim \ln K$, which is the D.I.S/twist two-like behaviour, we obtained other relations such as those of the form $E - K \sim K^{1/3}$, $E - K \sim K^3$. The first type of relation appeared previously for strings moving in Witten's QCD confining model [8]. The second type does not seem to have been previously studied. It is not clear which 'QCD-like' operator will be dual to these last two configurations. We must keep in mind that, in the gravity approximation, M-theory on a G_2 manifold is not dual to pure $\mathcal{N} = 1$ SYM. Indeed Kaluza-Klein and bulk modes are not decoupled from the $3 + 1$ gauge theory, a feature that seems to afflict any study involving D6 branes.

One might speculate that the logarithmic configurations we found on the G_2 spacetimes could be related to large lorentzian spin operators in field theory via Wilson lines. Wilson lines are closely related to the twist two operators of form (1.1), see for example [14] and references therein. The membrane configurations in question, called $\mathrm{IV}_{\mathbb{B}}$ and $\mathrm{IV}_{\mathbb{D}}$ above, form an infinite line in the noncompact directions. Some related comments were made in [16].

Finally, in the following appendices, we set up a formalism to study certain string configurations on warped AdS backgrounds, that are general confining backgrounds. We then end by explicitly checking non-supersymmetry of the membrane configurations we considered on G_2 backgrounds.

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A. Strings moving in warped AdS spaces

In this section we will consider strings rotating in the background geometry generated by a D4-D8 system. The objective is to study effects of warp factors on the rotating configurations. The subsection below takes a more general approach. The geometry is given in [55] and reads in Einstein frame,

$$\begin{aligned} ds^2 =& (\sin\xi)^{1/12}\Big[-dt^2\cosh^2\rho + d\rho^2 + \sinh^2\rho d\Omega_4^2 + \\ & + \frac{2}{g^2}\left(d\xi^2 + \frac{\cos^2\xi}{4}\left(d\theta^2 + d\phi^2 + d\psi^2 + 2\cos\theta d\phi d\psi\right)\right)\Big], \end{aligned} \tag{A.1}$$

where g is a constant and

$$d\Omega_4^2 = d\theta_1 + \cos^2\theta_1 d\theta_2 + \cos^2\theta_1\cos^2\theta_2 d\theta_3^2 + \cos^2\theta_1\cos^2\theta_2\cos^2\theta_3 d\theta_4^2\,, \tag{A.2}$$

and the matter fields are

$$e^{-6/5\phi} = \sin\xi\,, \qquad F_4 = \sin^{1/3}\xi\cos^3\xi \mathrm{vol}(S^4)\,. \tag{A.3}$$

We will consider here a configuration given by

$$\rho = \rho(\sigma)\,, \quad t = \kappa\tau\,, \quad \theta_4 = \omega\tau\,, \quad \xi = \xi(\sigma)\,, \quad \psi = \sqrt{2}\nu\tau\,, \quad \theta_i = 0\,, \quad \theta = 0\,. \tag{A.4}$$

Plug this configuration into the string action, and neglect the term that is proportional to the dilaton, since it is an α' correction and we are not considering in the metric (A.1) any stringy corrections. We have an action

$$S = \frac{2P}{g^2}\int d\sigma f(\xi)\left[\xi'^2 + \frac{g^2}{2}\rho'^2 + \frac{1}{2}\left(g^2\kappa^2\cosh^2\rho - \omega^2 g^2\sinh^2\rho\right) - \nu^2\cos^2\xi\right], \tag{A.5}$$

with $f(\xi) = \sin^{1/12}\xi(\sigma)$ and $'$ denoting $d/d\sigma$. The equations of motion are

$$\frac{d}{d\sigma}(f(\xi)\rho') - f(\xi)(\omega^2 - \kappa^2)\cosh\rho\sinh\rho = 0\,, \tag{A.6}$$

$$\begin{aligned} \frac{d}{d\sigma}(f(\xi)\xi') =& \frac{g^2}{4}\frac{df(\xi)}{d\xi}\left(\kappa^2\cosh^2\rho - \omega^2\sinh^2\rho - \frac{2\nu^2\cos^2\xi}{g^2} + \frac{2\xi'^2}{g^2} + \rho'^2\right) + \\ & + f(\xi)\nu^2\cos\xi\sin\xi\,, \end{aligned} \tag{A.7}$$

and the constraint reads

$$f(\xi)\left(\frac{2\xi'^2}{g^2} + \rho'^2 - \kappa^2\cosh^2\rho + \omega^2\sinh^2\rho + \frac{2\nu^2\cos^2\xi}{g^2}\right) = 0\,. \tag{A.8}$$

One can check that the derivative of the constraint can be split up to give the second order equations of motion.

We can consider special cases of the previous configurations. In the case where the warping angle ξ is taken to be a constant $\xi_* = \pi/2$. Note that this specific value is needed to solve the equations of motion. There is no R-charge in this configuration. The computations are very similar to the cases analysed in [1] and we get the same result with operators satisfying $E - S \sim \ln S$. Besides, one can consider the case in which the coordinate $\rho = \rho_* = 0$ is constant, again with a value specified by the equations of motion. In this case, the equations of motion reduce to

$$\xi'' = \nu^2 \cos\xi \sin\xi \,, \qquad \xi'^2 + \nu^2 \cos^2 \xi = \frac{g^4}{4}\kappa^2 \,, \tag{A.9}$$

The turning point will be

$$\cos^2 \xi_0 = \frac{g^4}{4\nu^2}\kappa^2 \,. \tag{A.10}$$

We can compute the energy and angular momentum for this configuration to be given by

$$E = \frac{2}{g^2} P\kappa \int_0^{\xi} \frac{\sin^{1/12}\xi}{\sqrt{\frac{g^4}{4\nu^2}\kappa^2 - \nu^2\cos^2\xi}} d\xi \,, \tag{A.11}$$

$$J = \frac{2}{g^2} P\nu \int_0^{\xi} \frac{\sin^{1/12}\xi}{\sqrt{\frac{g^4}{4\nu^2}\kappa^2 - \nu^2\cos^2\xi}} \cos^2\xi d\xi \,. \tag{A.12}$$

As we have done in previous sections, we expand the integrals above to find the relation between energy, spin and angular momentum for long and short strings. After doing the expansion, we notice that the energy and angular momentum do not diverge for long strings. This is a new type of behaviour. Even though this geometry is very similar to the one described in section 4.1 of the paper [1], we have here a warping factor. We get that the relation for long strings is of the form

$$E - J = k + \cdots \,, \tag{A.13}$$

where k is a numerical constant.

A.1 Strings moving in a general background

Consider now the motion of strings in backgrounds of the form,

$$ds_{10}^2 = f(r)\left[-dt^2 + d\rho^2 + \rho^2 d\Omega_2^2(\theta,\phi)\right] + dr^2 + \cdots \tag{A.14}$$

where the "$\cdots$" can be whatever one wants, the string will not be moving in these directions. Backgrounds of this form are interesting since they have the general form of gravity duals to gauge theories in $3+1$ dimensions with a low number of supersymmetries, and that may exhibit confinement. Consider a string configuration that could be interpreted as a spinning string in the $3+1$ manifold

$$r = r(\sigma)\,, \quad t = \kappa\tau\,, \quad \phi = \omega\tau\,, \quad \rho = \rho(\sigma)\,. \tag{A.15}$$

The Polyakov action in this case is,

$$I = \frac{1}{2\pi\alpha'} \int d\sigma \left[r'^2 + f(r)(\rho'^2 + \kappa^2 - \rho^2\omega^2) \right], \tag{A.16}$$

and the constraint is,

$$f(r)(\rho'^2 - \kappa^2 + \rho^2\omega^2) + r'^2 = 0\,. \tag{A.17}$$

We can see that the derivative of the constraint will have the form

$$2r' \left(r'' - \frac{1}{2}\frac{df}{dr}(\kappa^2 + \rho'^2 - \omega^2\rho^2) \right) + 2\rho' \left(f(r)\rho'' + f(r)\omega^2\rho + \frac{df}{dr} r'\rho' \right) = 0\,, \tag{A.18}$$

where terms inside the parenthesis will be the equations of motion derived from (A.16). Integrating the equations of motion, we will obtain a relation between the variables, that we can substitute into the original action and follow the procedure in previous sections of the paper. This seems like an interesting direction to investigate in the future. It seems possible that one might obtain logarithms in these types of configurations.

B. Conditions for supersymmetry of rotating membranes

We do not expect our configurations to be supersymmetric given the time dependence and the minimally supersymmetric background. However, for completeness, we check this explicitly.

Let ϵ generate a supersymmetry of the background spacetime metric, i.e. it is a Killing spinor. This supersymmetry will be preserved by the worldsheet if [56, 39, 57]

$$\Gamma_{M2}\epsilon = \epsilon\,, \tag{B.1}$$

where

$$\Gamma_{M2} = \frac{1}{\sqrt{-\det\gamma}} \frac{1}{3!} \epsilon^{ijk} \partial_i X^\mu \partial_j X^\nu \partial_k X^\sigma \Gamma_{\mu\nu\sigma}\,. \tag{B.2}$$

Where $\Gamma_{\mu\nu\sigma}$ is the standard antisymmetric combination of eleven dimensional Dirac matrices, and γ_{ij} is the induced metric on the worldsheet.

This formula is particularly easy to apply to the membrane configurations on G_2 backgrounds. For the $\mathbb{B}_7$ metrics, the parallel spinor [30] has constant coefficients and satisfies three projection conditions

$$\Gamma_{2635}\epsilon = \epsilon\,, \qquad \Gamma_{1634}\epsilon = \epsilon\,, \qquad \Gamma_{6201}\epsilon = \epsilon, \tag{B.3}$$

using tangent space indices. The orthonormal frame on the G_2 is

$$\begin{aligned} e^0 &= dr\,, & e^1 &= a(\Sigma_1 - \sigma_1)\,, & e^2 &= a(\Sigma_2 - \sigma_2)\,, & e^3 &= d(\Sigma_3 - \sigma_3)\,, \\ e^4 &= b(\Sigma_1 + \sigma_1)\,, & e^5 &= b(\Sigma_2 + \sigma_2)\,, & e^6 &= d(\Sigma_3 + \sigma_3)\,. \end{aligned} \tag{B.4}$$

From here one uses the membrane configurations of table 1 to calculate Γ_{M2}. For example, for the $\mathrm{I}_{\mathbb{B}}$ configuration one obtains

$$\Gamma_{\mathrm{I}_{\mathbb{B}}} = \frac{\left[\omega\Gamma_{\phi_3} + \nu_2\Gamma_{\phi_4} + \kappa\Gamma_t\right]\Gamma_r\Gamma_{\psi_3}}{c/N\sqrt{\kappa^2 - b^2\omega^2 - a^2\nu_2^2}}\,, \tag{B.5}$$

where

$$\begin{aligned}
\Gamma_{\phi_3} &= b\sin\frac{\psi_3}{2}\Gamma_4 + b\cos\frac{\psi_3}{2}\Gamma_5\,,\\
\Gamma_{\phi_4} &= a\sin\frac{\psi_3}{2}\Gamma_1 + a\cos\frac{\psi_3}{2}\Gamma_2\,,\\
\Gamma_{\psi_3} &= \frac{c}{N}\Gamma_6\,,\\
\Gamma_r &= \Gamma_0\,, \qquad \Gamma_t = \Gamma_{\hat{t}}\,.
\end{aligned} \tag{B.6}$$

The matrix (B.5) is easily seen not to commute or anticommute with the projectors of equation (B.3) and therefore no supersymmetries are preserved. The same will be the case for the $\mathrm{II}_{\mathbb{B}}$, $\mathrm{III}_{\mathbb{B}}$ and $\mathrm{IV}_{\mathbb{B}}$ configurations.

The $\mathbb{D}_7$ cases are a little more complicated because the parallel spinor [49] does not have constant coefficients. However, it will be sufficient for us to know that the parallel spinor satisfies

$$\Gamma_{2536}\epsilon = \epsilon\,, \tag{B.7}$$

where we are using tangent space indices and the vielbeins are

$$\begin{aligned}
e^0 &= dr\,, \quad & e^1 &= a(\Sigma_1 + g\sigma_1)\,, \quad & e^2 &= a(\Sigma_2 + g\sigma_2)\,, \quad & e^3 &= c(\Sigma_3 + g_3\sigma_3)\,,\\
e^4 &= b\sigma_1\,, \quad & e^5 &= b\sigma_2\,, \quad & e^6 &= f\sigma_3\,.
\end{aligned} \tag{B.8}$$

One then calculates Γ_{M2} using table 2. For the $\mathrm{I}_{\mathbb{D}}$ configuration, for example, one obtains

$$\Gamma_{\mathrm{I}_{\mathbb{D}}} = \frac{\left[\omega_1\Gamma_{\phi_1} + \nu\Gamma_{\phi_2} + \kappa\Gamma_t\right]\Gamma_r\Gamma_{\psi_3}}{C/N\sqrt{\kappa^2 - \omega_1^2B^2 - \nu^2A^2}}\,, \tag{B.9}$$

where

$$\begin{aligned}
\Gamma_{\phi_1} &= \frac{ag}{\sqrt{2}}\left[\left(\sin\frac{\psi_3}{2} + \cos\frac{\psi_3}{2}\right)\Gamma_1 + \left(\cos\frac{\psi_3}{2} - \sin\frac{\psi_3}{2}\right)\Gamma_2\right] +\\
&\quad + \frac{b}{\sqrt{2}}\left[\left(\cos\frac{\psi_3}{2} + \cos\frac{\psi_3}{2}\right)\Gamma_4 + \left(\cos\frac{\psi_3}{2} - \sin\frac{\psi_3}{2}\right)\Gamma_5\right],\\
\Gamma_{\phi_2} &= a\left(\sin\frac{\psi_3}{2} + \cos\frac{\psi_3}{2}\right)\Gamma_1 + a\left(\sin\frac{\psi_3}{2} - \cos\frac{\psi_3}{2}\right)\Gamma_2\,,\\
\Gamma_{\psi_3} &= \frac{cg^2}{N}\Gamma_2 + \frac{f}{2N}\Gamma_6\,,\\
\Gamma_t &= \Gamma_{\hat{t}}\,, \qquad \Gamma_r = \Gamma_0\,.
\end{aligned} \tag{B.10}$$

One can now see that $\Gamma_{\mathrm{I}_{\mathbb{D}}}$ and Γ_{2536} do not commute or anticommute and therefore no supersymmetry is preserved. It is easy to check that the same occurs for the other configurations, $\mathrm{II}_{\mathbb{D}}$, $\mathrm{III}_{\mathbb{D}}$ and $\mathrm{IV}_{\mathbb{D}}$. Thus, as expected, none of our configurations are supersymmetric.

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